Equilibrium and Dynamics of Large Stochastic Networks

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Contents

1 Introduction **3**

2 A Mobile Network **43**

1. Introduction

Stochastic Network

A Distributed System

— Set of autonomous nodes

— Local Dynamics: Same algorithm at each node to

- Transmit data; Establish connections; Retrieve information; . . .

Stochastic Network

A Distributed System

— Set of **autonomous** nodes

— **Local Dynamics:** Same algorithm at each node to

- Transmit data; Establish connections; Retrieve information; . . .

General Problem:

Infer global behavior of the network from local specifications.

Circuit Switching Networks

Circuit Switching Networks

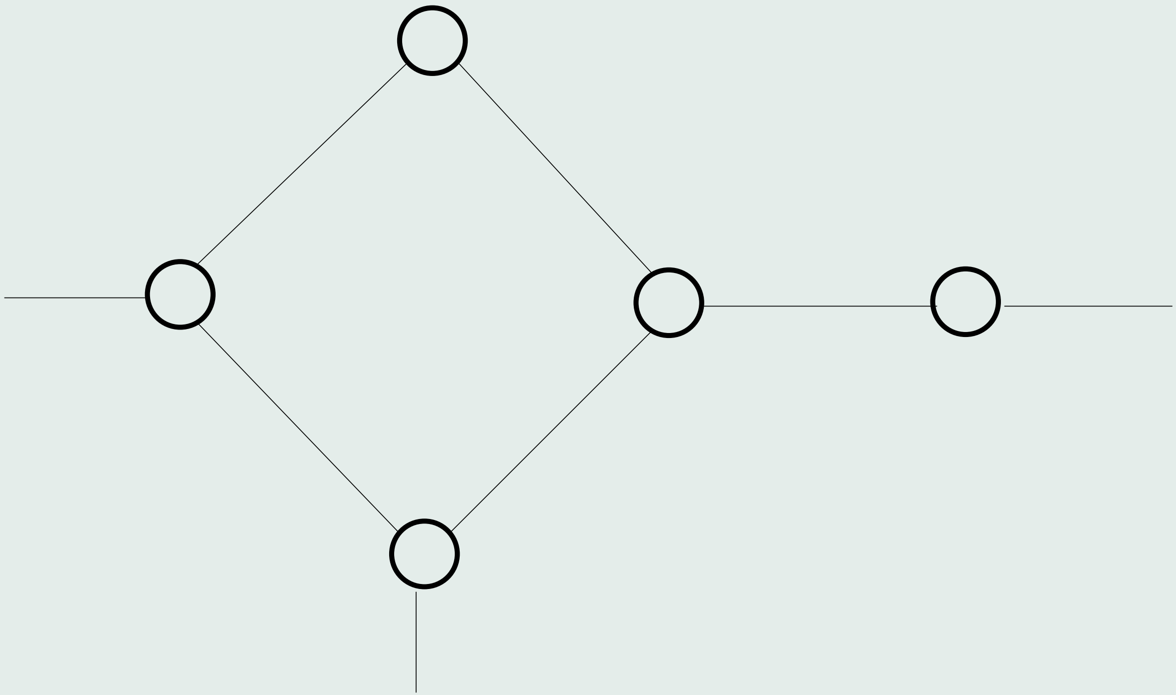
— Network: a set of links.

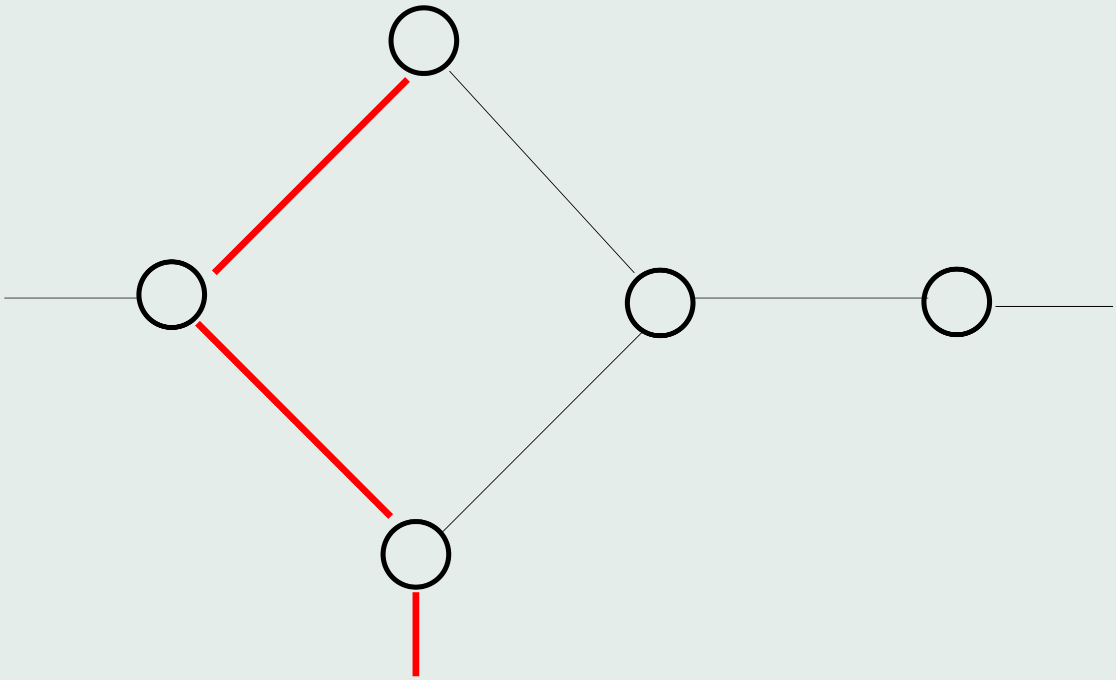
— A connection requires simultaneously a subset of links (a route).

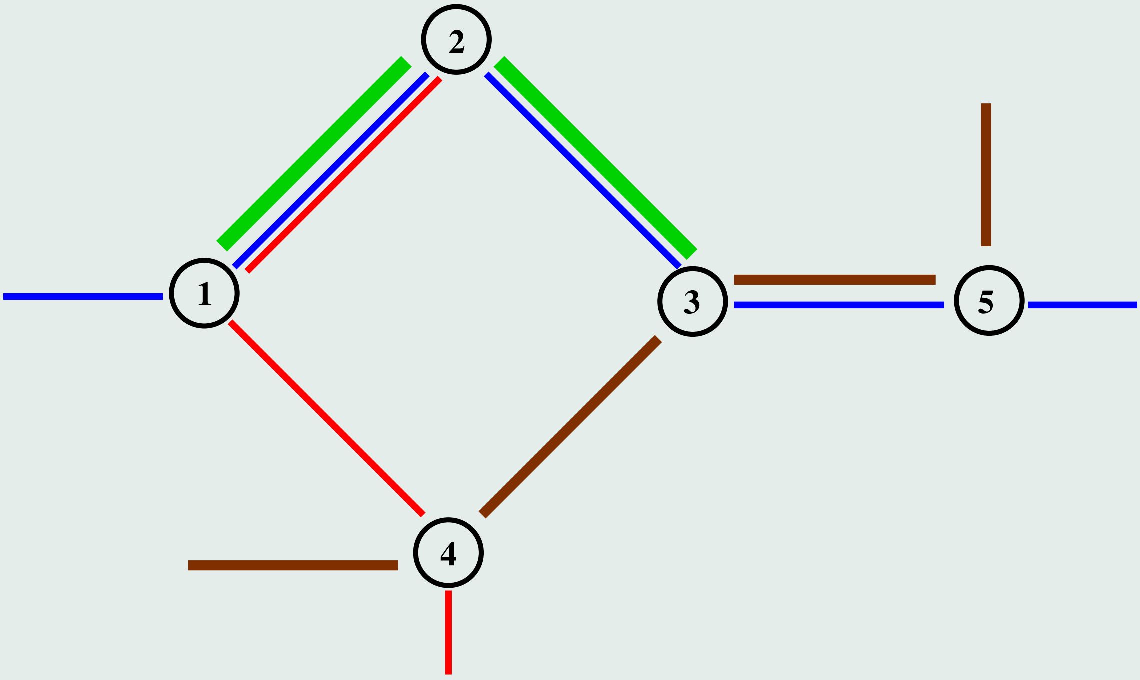
— Different classes of connexions.

— A link has a finite capacity.

— Capacity constraint: a connexion whose route requires a saturated link is rejected.







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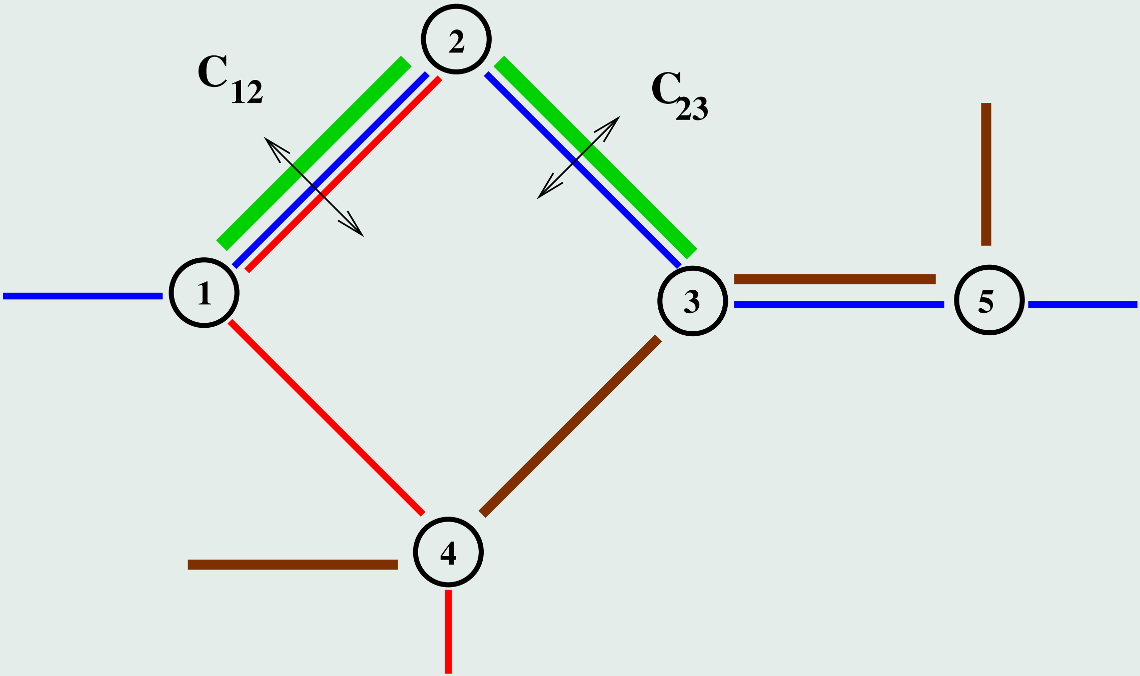

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5

Circuit Switching Networks: History

— Oldest mathematical model of networks

— Reversible Markov processes

— Close to classical particle systems model
Voter model.

Huge literature. See Kelly's survey (1991).

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Analysis of Partition Function.

Circuit Switching Networks: History

— Oldest mathematical model of networks

— Reversible Markov processes

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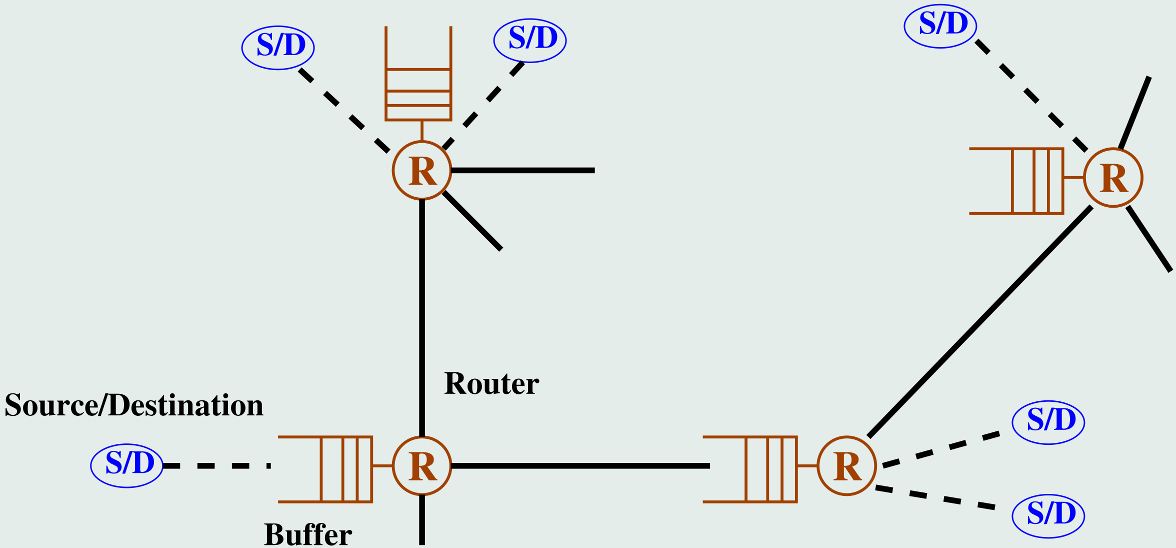
Huge literature. See Kelly's survey (1991).

— Equilibrium: well understood
Analysis of Partition Function.

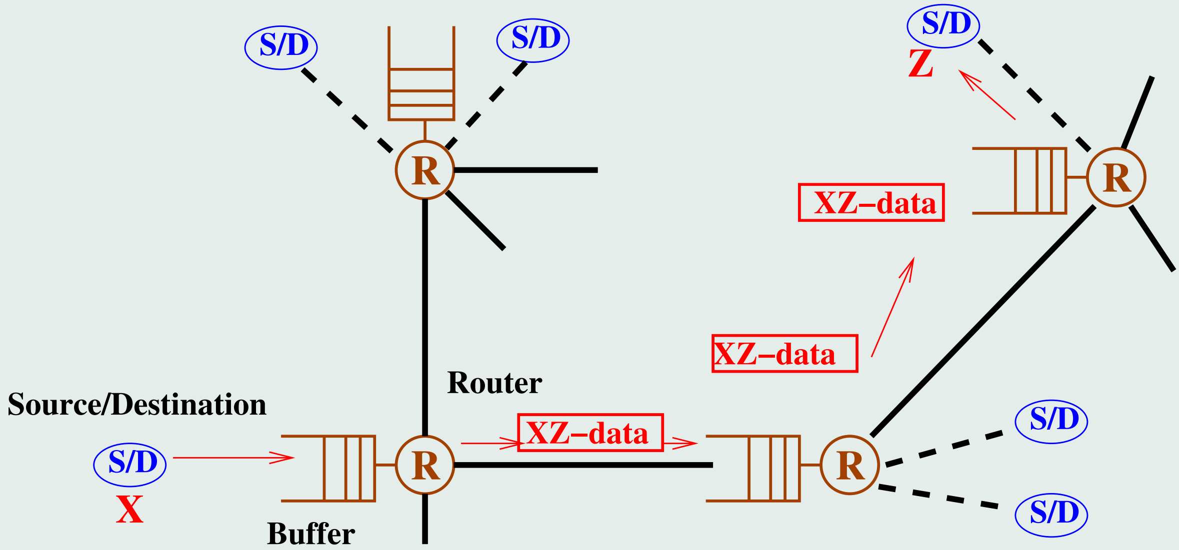
— Transient/Dynamical behavior is less known.

Packet Switching Networks

Packet Switching Networks



Packet Switching Networks



Packet Switching Networks

Throughput $W(t)$ of a connection

— No congestion: Increases linearly

$$\frac{d}{dt}W(t) = a$$

— Congestion at some (random) time t

$$W(t+) = bW(t-), \quad 0 < b < 1.$$

A self-adaptive algorithm to avoid congestion.

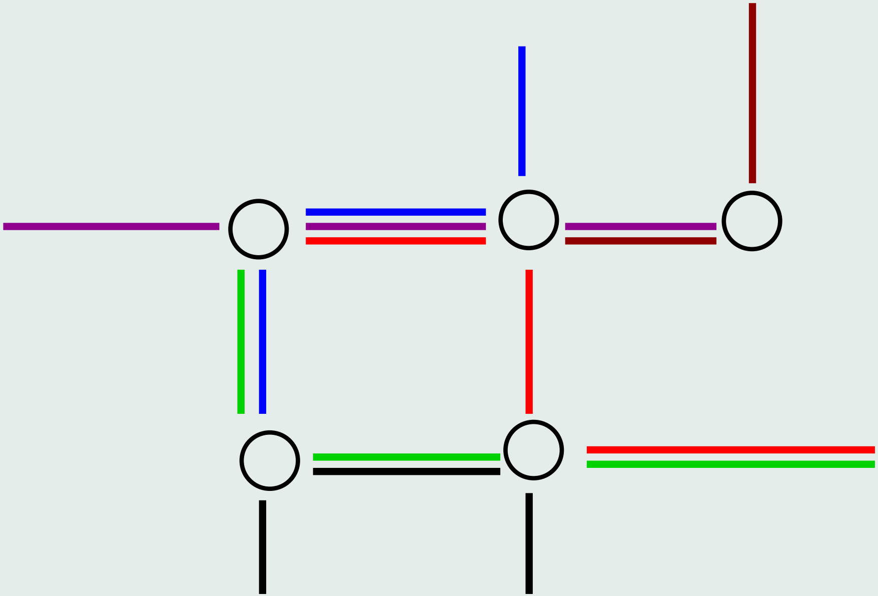
Packet Switching Networks: Maths

For a network with more than ONE node

— **Huge literature**

— **Few rigorous results**
simulations, approximations, . . .

Packet Switching Networks: A Fluid Picture



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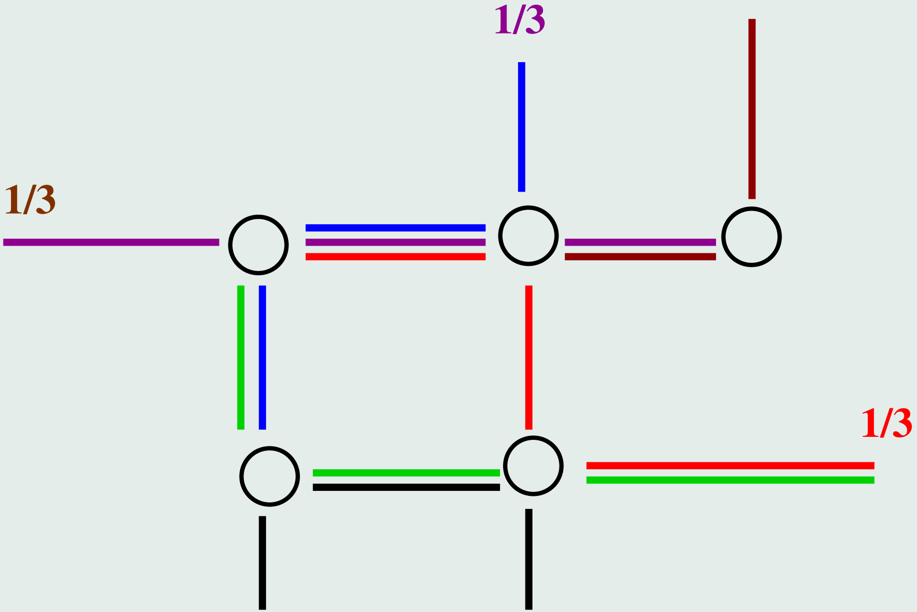

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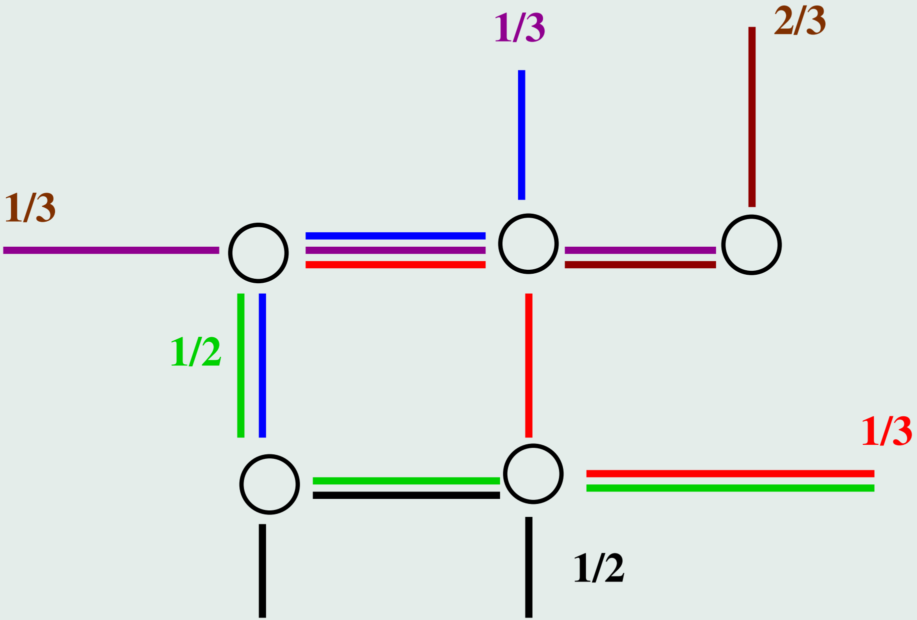


Two approaches (I)

Fluid Level

— If x_r routes of class $r \in \mathcal{R}$, each receive throughput λ_r^0, such that (λ_r^0) achieves the maximum

$$\max_{\lambda \in \mathcal{C}} \sum_{r \in \mathcal{R}} x_r U(\lambda_r / x_r)$$

U utility function.

— Massoulié and Roberts (1999)
Kelly and Williams (2004), Massoulié (2007).

Two approaches (II)

Packet Level

— A connection: a Markov process with generator

$$\Omega(f)(x) = af'(x) + b(x)(f(rx) - f(x)),$$

Ott *et al.* (1996), Dumas *et al.* (2002), . . .

— Coexistence of different classes of connections
Graham and Robert (2008).

Wireless Networks

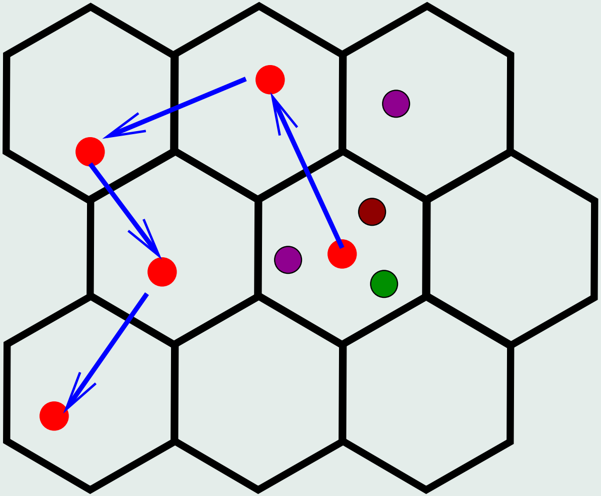
Wireless Networks

— **Network: a set of cells**

— **Mobiles move among cells**

— **Interaction**
Mobiles of the same cell "share" its capacity.

Wireless Networks



Wireless Networks

Problems

— Capacity of the network

— Delay of transmissions

Wireless Networks

Problems

— Capacity of the network

— Delay of transmissions

Literature

Study of processor-sharing policies

Roberts and co-authors, Borst, . . .

Mobile Networks with no Infrastructure

Mobile Networks without Infrastructure

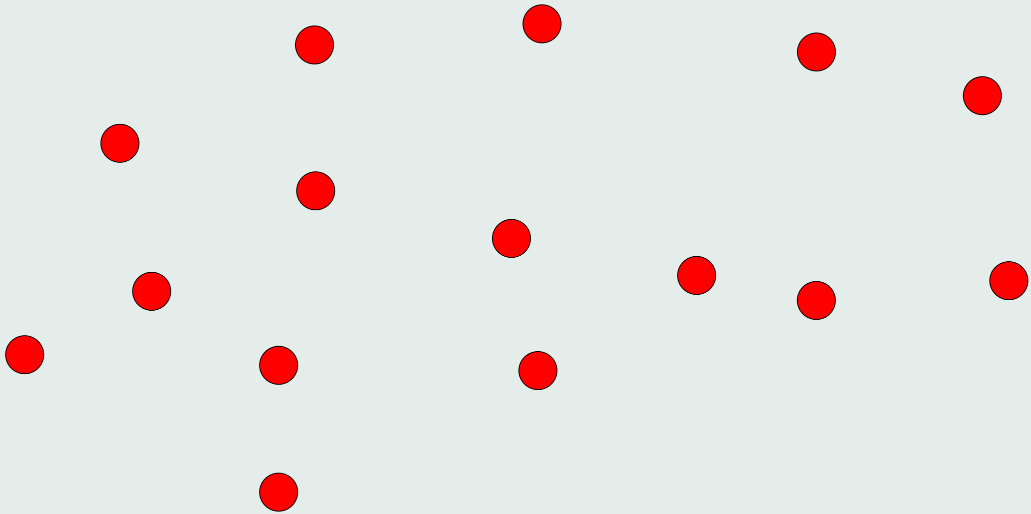
— **No fixed network.**

— **Wireless mobiles.**

— **A node "can see" nodes within some area.**

— **Link: two nodes that can see each other.**

Mobile Networks without Infrastructure



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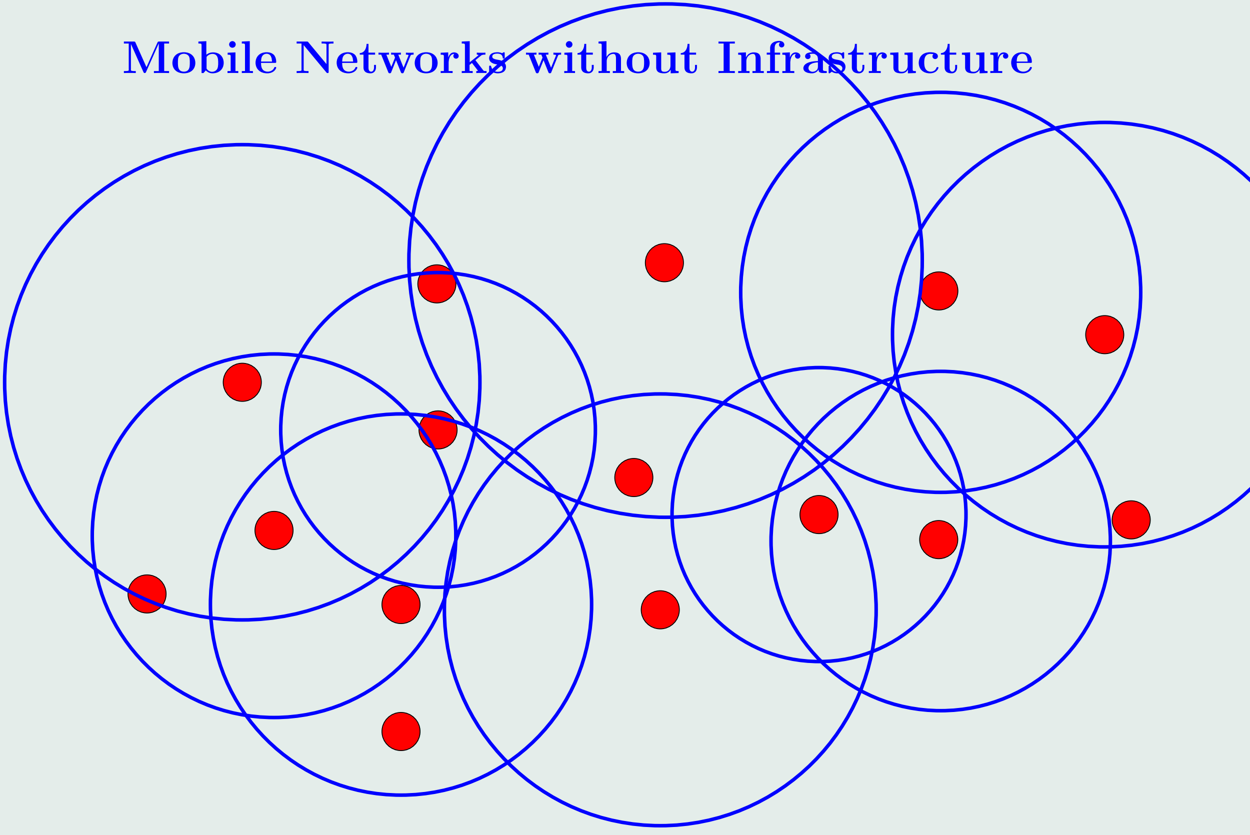

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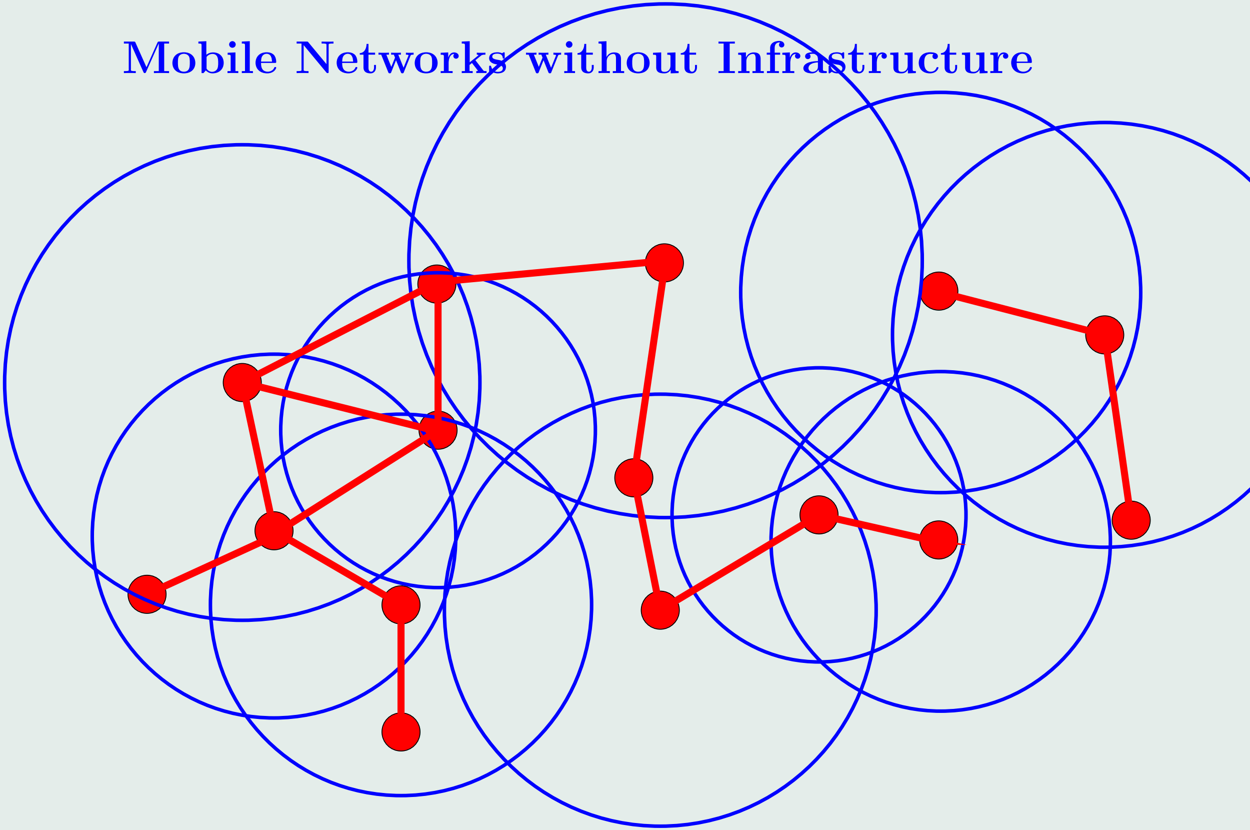


Mobile Networks without Infrastructure

— **Connectivity of the network**

— **Lengths of paths**

— **Impact of mobility**

Mobile Networks without Infrastructure

— **Connectivity of the network**

— **Lengths of paths**

— **Impact of mobility**

Literature
Gupta and Kumar, . . .

Virtual Networks

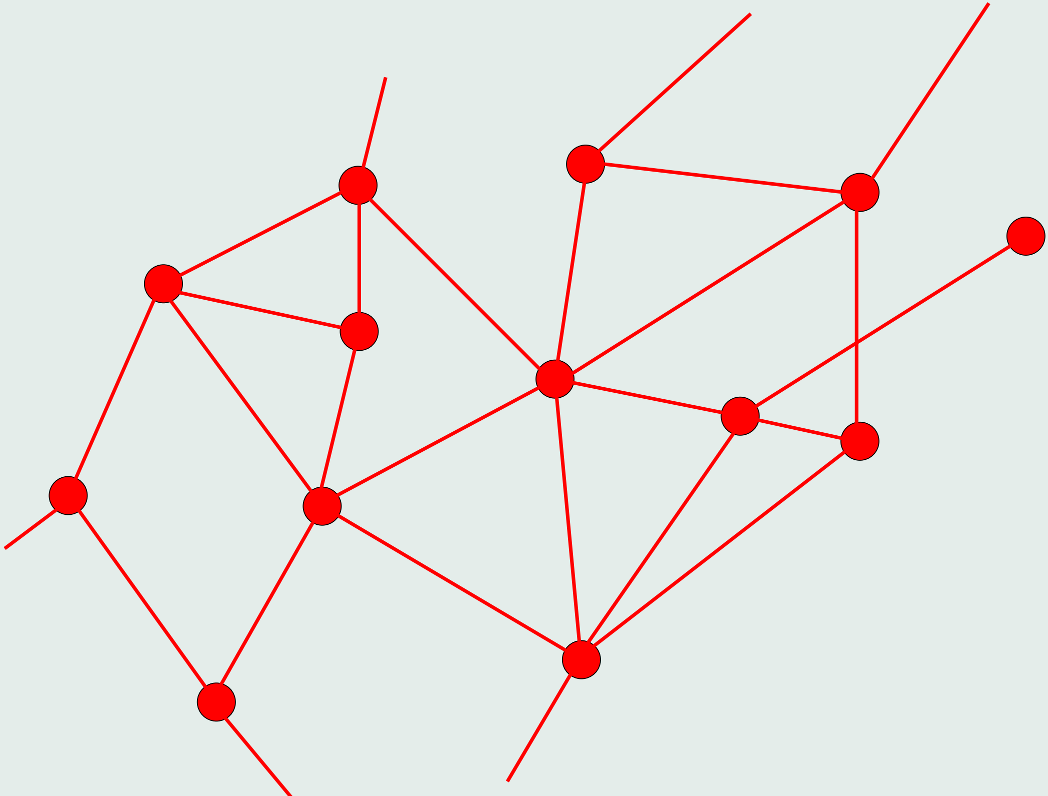
Virtual Networks

A subset of nodes of a real network organized in a network

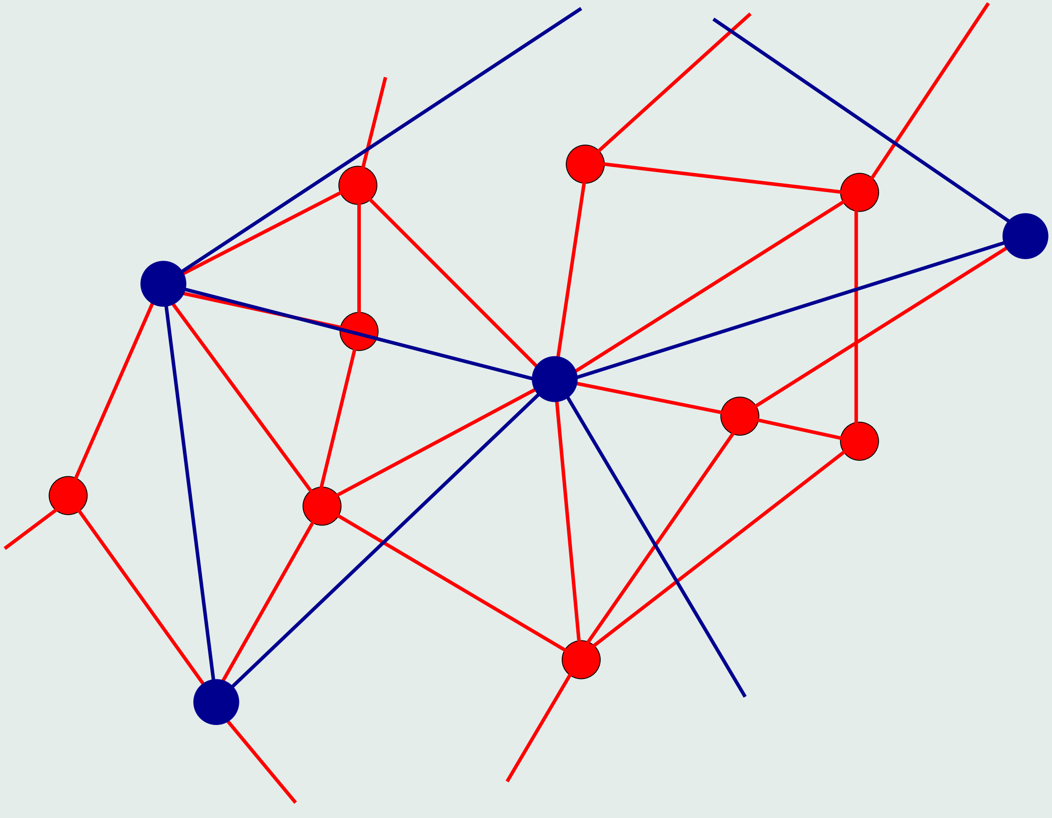
— **virtual links**

— **transmit, retrieve information, . . .**

“Real” Network



Virtual Networks



Virtual Networks

Problems

— How fast the information is transmitted ?

— Impact on the “real” network.

— Algorithms: e-donkey, bittorrent ?

Virtual Networks

Problems

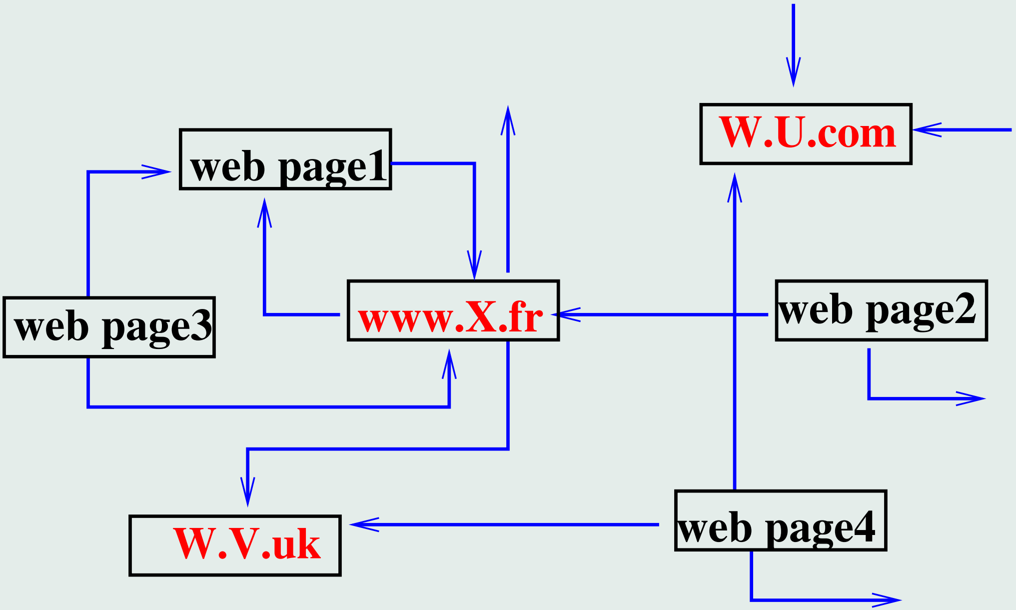
— How **fast** the information is transmitted ?

— **Impact** on the "real" network.

— **Algorithms:** e-donkey, bittorrent ?

Literature

Epidemic models, Random Graphs, . . .



A Virtual Network : WEB

Maths: Scalings are a usefool tool

Maths: Markov Jump Processes

— $X(t) = (X_j(t), 1 \leq j \leq N) \in (\mathbb{N}^d)^N$

— No reversibility properties.

— No product form for invariant measure.

— Exponential distributions.

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Exceptions: Bramson, Rybko and Shlosman.

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A normal network

A (very) large number of nodes.

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A normal network

A (very) large number of nodes.

$\Rightarrow$ scalings and limit theorems

Markov Jump Processes: Scalings

Fluid Scalings:

$$(x(t)) \text{''}=\text{''} \lim_{C=\|X(0)\|\to+\infty} \left(\frac{X(Ct)}{C}\right)$$

Heavy Traffic Scalings: λ input rate

$$(x(t)) \text{''}=\text{''} \lim_{\lambda\to+\infty} \left(\frac{X(t)}{\lambda}\right)$$

...

A first order description of the network

Markov Jump Processes: Mean-Field Scalings

Number N of nodes/connections goes to infinity

— Some symmetry conditions

— Asymptotic behavior $(x(t))$ of a "typical" node/connection ?

Convergence results and study of master equations

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A case by case approach ?

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Number N of nodes/connections goes to infinity

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Convergence results and study of master equations

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Early studies
Dobrushin and co-authors, Graham and Méléard

Markov Jump Processes: Scalings

Asymptotic (random) dynamical system $(x(t))$

— **Fixed points or Equilibrium Distributions**

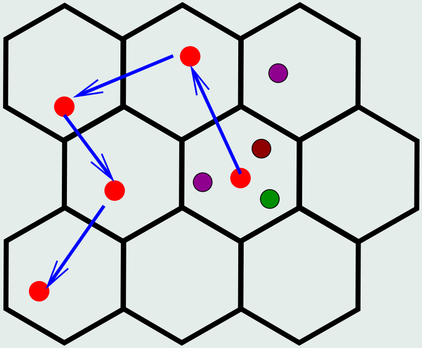
— **Stability properties**

— **Properties of the original Markov process ?**

2. A Mobile Network

A Network of Mobiles

joint work with Antunes, Fricker, Tibi — Ann. Proba. (2008)



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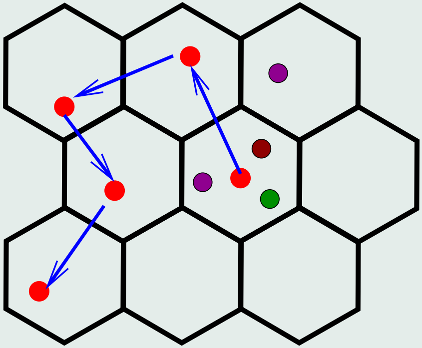

— Mobile M requires bandwidth a in a cell

— Total bandwidth available in a cell: C

— Rejection if M cannot get a in a cell

A Stochastic Model

Class k customers, $1 \leq k \leq K$,

— Arrive at rate λ_k
Poisson process;

— Require capacity A_k

— Leave the network at rate μ_k (can be 0)
Exponential Distribution

— Move from a cell to another at rate γ_k
Exponential Distribution

— Move uniformly among nodes.

A Markov Process

$X_{i,k}(t)$: number of class k customers at node $i \in \{1, \ldots, N\}$ at time t.

Capacity constraint:

$$\sum_{k=1}^{K} A_k X_{i,k}(t) \le C, \quad \forall i \in \{1, \ldots, N\}.$$

$X(t) = (X_{i,k}(t), 1 \le i \le N, 1 \le k \le K)$

Markov process

A Markov Process

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ergodic Markov process (finite state space).

A Markov Process (II)

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Scalings Analyzed:

1. Kelly's Limiting Regime
2. Thermodynamic Limit Regime.

State space for a node

$$\mathcal{X} = \{ n = (n_k) \in \mathbb{N}^K : A_1 n_1 + \cdots + A_K n_K \leq C \}$$

Markovian Representation

$$X^N(t) = (X_i^N(t), 1 \le i \le N)$$

with $X_i^N(t) = (X_{i,k}^N(t), 1 \le k \le K)$

For $n = (n_k, 1 \le k \le K) \in \mathcal{X}$

$$Y_n^N(t) \stackrel{\text{def.}}{=} \frac{1}{N} \#\{1 \le i \le N : X_i^N(t) = n\}$$

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Consequence of Symmetry:

$Y^N(t) = (Y_n^N(t), n \in \mathcal{X})$ is a Markov process.

Empirical Distribution

Another Formulation:

$Y_n^N(t)$ **: proba that a random node of the network is equal to** $n=(n_k)$ **at time** t**.**

Empirical Distribution

Another Formulation:

$Y_n^N(t)$ **:** **proba that a random node of the network is equal to** $n{=}(n_k)$ **at time** t.

$Y^N(t) \in \mathcal{P}(\mathcal{X})$,

$\mathcal{P}(\mathcal{X})$: **set of probability distributions on** $\mathcal{X}$.

Convergence of Empirical Distributions

Theorem

For $y \in \mathcal{P}(\mathcal{X})$, if $Y^N(0) \to y$ then

$$(Y^N(t), t \geq 0) \stackrel{\text{dist.}}{\longrightarrow} (y(t), t \geq 0),$$

the solution of

$$\frac{dy}{dt}(t) = V(y(t)), \quad y(0) = y,$$

$V(y) = (V_n(y), n \in \mathcal{X})$ vector field on $\mathcal{P}(\mathcal{X})$.

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$(y(t))$ dynamical system on $\mathcal{P}(\mathcal{X})$.

The Vector Field

$$V_n(y) = \sum_{k=1}^{K} v_{n,k}(y)$$

where $v_{n,k}(y)$ is given by

$$\begin{aligned}(\lambda_k + \gamma_k \langle \mathbb{I}_k, y\rangle) \left(y_{n-f_k} 1_{\{n_k \geq 1\}} - y_n 1_{\{n+f_k \in \mathcal{X}\}} \right) \\ + \ (\gamma_k + \mu_k) \left((n_k + 1) y_{n+f_k} 1_{\{n+f_k \in \mathcal{X}\}} - n_k y_n \right)\end{aligned}$$

---

with $\langle \mathbb{I}_k, y\rangle = \sum_{m \in \mathcal{X}} m_k y_m$

f_k is the kth unit vector of $\mathbb{R}^K$.

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with $\langle \mathbb{I}_k, y \rangle = \sum_{m \in \mathcal{X}} m_k y_m$
f_k is the kth unit vector of $\mathbb{R}^K$.

Quadratic dependence on y.

Equilibrium Points: $\{y \in \mathcal{P}(\mathcal{X}) : V(y) = 0\}$

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Proposition

If $y \in \mathcal{P}(\mathcal{X})$ such that $V(y) = 0$, then $y = \nu_\rho$,

$$\nu_\rho(n) = \frac{1}{Z(\rho)} \frac{\rho^n}{n!}, \quad n = (n_k) \in \mathcal{X},$$

$\rho = (\rho_k) \in \mathbb{R}_+^K$

with $Z(\rho)$ normalization constant of ν_ρ
and $\rho^n/n! = \prod_{k=1}^K \rho_k^{n_k}/n_k!$ for $n = (n_k) \in \mathcal{X}$.

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$\rho = (\rho_k) \in \mathbb{R}_+^K$ satisfies fixed point equations

$$\lambda_k = \rho_k \left(\mu_k + \gamma_k \frac{\sum_{n : n + f_k \notin \mathcal{X}} \rho^n / n!}{\sum_{n \in \mathcal{X}} \rho^n / n!} \right), 1 \leq k \leq K.$$

---

with $Z(\rho)$ normalization constant of ν_ρ
and $\rho^n / n! = \prod_{k=1}^K \rho_k^{n_k} / n_k!$ for $n = (n_k) \in \mathcal{X}$.

Existence

Proposition

y is an equilibrium point if and only if it is a fixed point of the function

$$\begin{aligned} \mathcal{P}(\mathcal{X}) &\longrightarrow \mathcal{P}(\mathcal{X}) \\ y &\longrightarrow \nu_{\rho(y)}, \end{aligned}$$

with $\rho(y) = (\rho_k(y))$ and

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Brouwer's Theorem $\Rightarrow$ **existence.**

A Dimension Reduction ?

Equation $V(y) = \mathbf{0}$ reduced to K equations

$\mathcal{P}(\mathcal{X}) \Rightarrow$ **Manifold of** $\mathbb{R}_+^K$

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Equation $V(y) = 0$ reduced to K equations

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1. $\not\Rightarrow$ **dimension reduction**
   for dynamical systems:
   $y(t) \neq \nu_{\rho(t)}$ **with** $(\rho(t))$ **dynamical system on** $\mathbb{R}_+^K$.

A Dimension Reduction ?

Equation $V(y) = 0$ reduced to K equations

$\mathcal{P}(\mathcal{X}) \Rightarrow$ Manifold of $\mathbb{R}_+^K$

1. $\not\Rightarrow$ dimension reduction
   for dynamical systems:
   $y(t) \neq \nu_{\rho(t)}$ with $(\rho(t))$ dynamical system on $\mathbb{R}_+^K$.
2. $\Rightarrow$ no indication on
   Stability of the equilibrium points

Literature

Gibbens, Hunt and Kelly (1990),
Marbukh (1993)

— Loss networks with alternative routing.

— One class of customers.

— Similar setting and description of equilibrium points.

— Analysis through approximations and numerical estimations.

One class of customers

Proposition

With one class of customers there is a unique equilibrium point.

An Energy Function on $\mathcal{P}(\mathcal{X})$

For $y \in \mathcal{P}(\mathcal{X})$,

$$g(y) = \sum_{n \in \mathcal{X}} y_n \log(n! y_n) - \sum_{k=1}^{K} \int_0^{\langle \mathbb{I}_k, y \rangle} \log \frac{\lambda_k + \gamma_k x}{\mu_k + \gamma_k} \, dx.$$

with $\langle \mathbb{I}_k, y \rangle = \sum_{m \in \mathcal{X}} m_k y_m$.

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with $\langle \mathbb{I}_k, y \rangle = \sum_{m \in \mathcal{X}} m_k y_m$.

Theorem

g decreases along the trajectory of $(y(t))$

$$\frac{dg \circ y}{dt}(t) = \langle V(y(t)), \nabla g(y(t)) \rangle \leq 0, \quad \forall t \geq 0$$

An “Energy” Function on $\mathbb{R}^K$

$$\phi(\rho) = -\log Z(\rho) + \sum_{k=1}^{K} (\beta_k \rho_k - \alpha_k \log(\rho_k))$$

with

$$\alpha_k = \lambda_k/\gamma_k, \quad \beta_k = (\gamma_k + \mu_k)/\gamma_k$$

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ν_ρ *equilibrium point of* $(y(t))$ *iff* $\nabla \phi(\rho) = 0$.

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ν_ρ *equilibrium point of* $(y(t))$ *iff* $\nabla \phi(\rho) = 0$.

Extrema of g and ϕ are in correspondence.

Complete Correspondence of Extrema

Theorem

1. ρ local minimum of ϕ iff ν_ρ local min. of g.
2. If ρ saddle point of ϕ,
   then ν_ρ saddle point of g.

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Proof: Comparison of quadratic forms.

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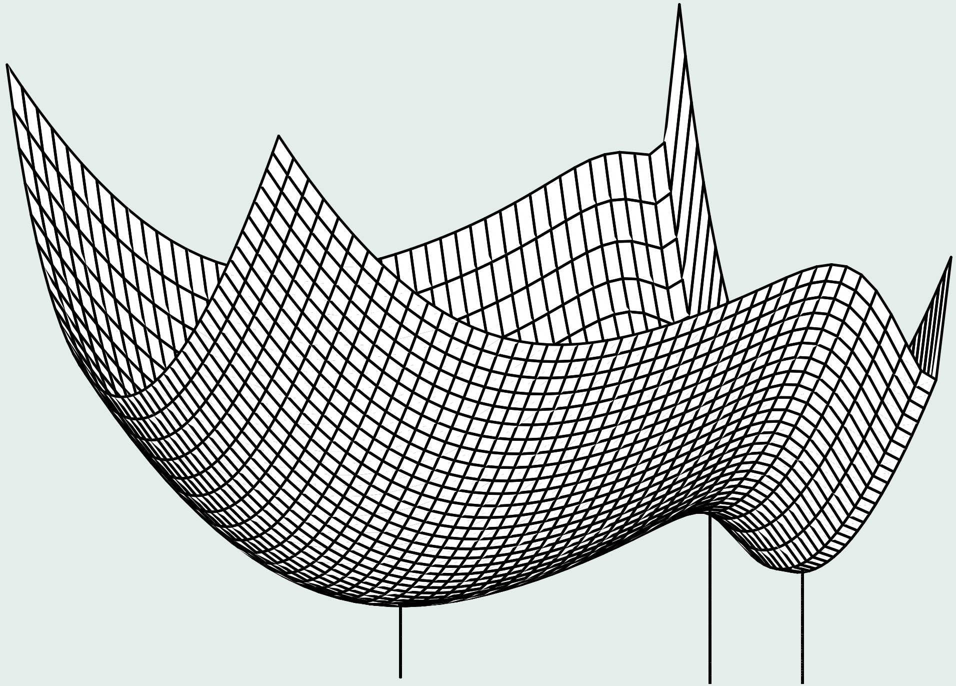
Dimension reduction
achieved through energy functions.

Bistability

Network with two classes of customers:
If $A_1 = 1$, $A_2 = C$, $\gamma_1 = \gamma_2 = 1$, $\mu_1 = \mu_2 = 0$, $\exists \lambda_1, \lambda_2 \in \mathbb{R}_+$ so that for C large, ϕ has at least one saddle point and two local minima.

$\Rightarrow \exists$ network with two stable equilibrium points.

Function ϕ for counter-example



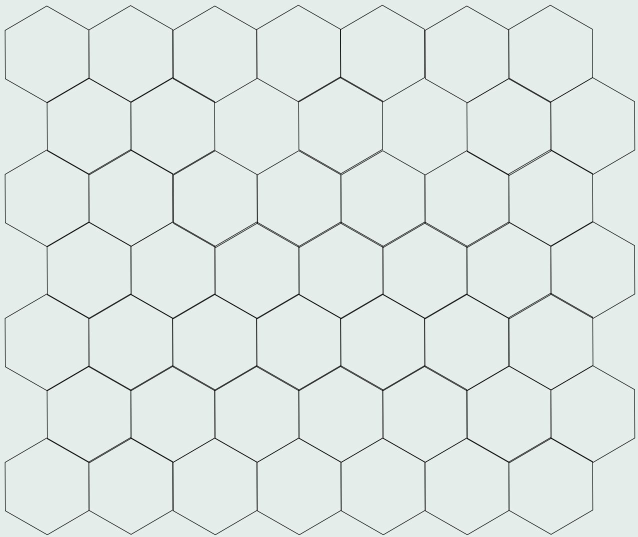
$K=2$ — $\dim P(\mathcal{X})=22$

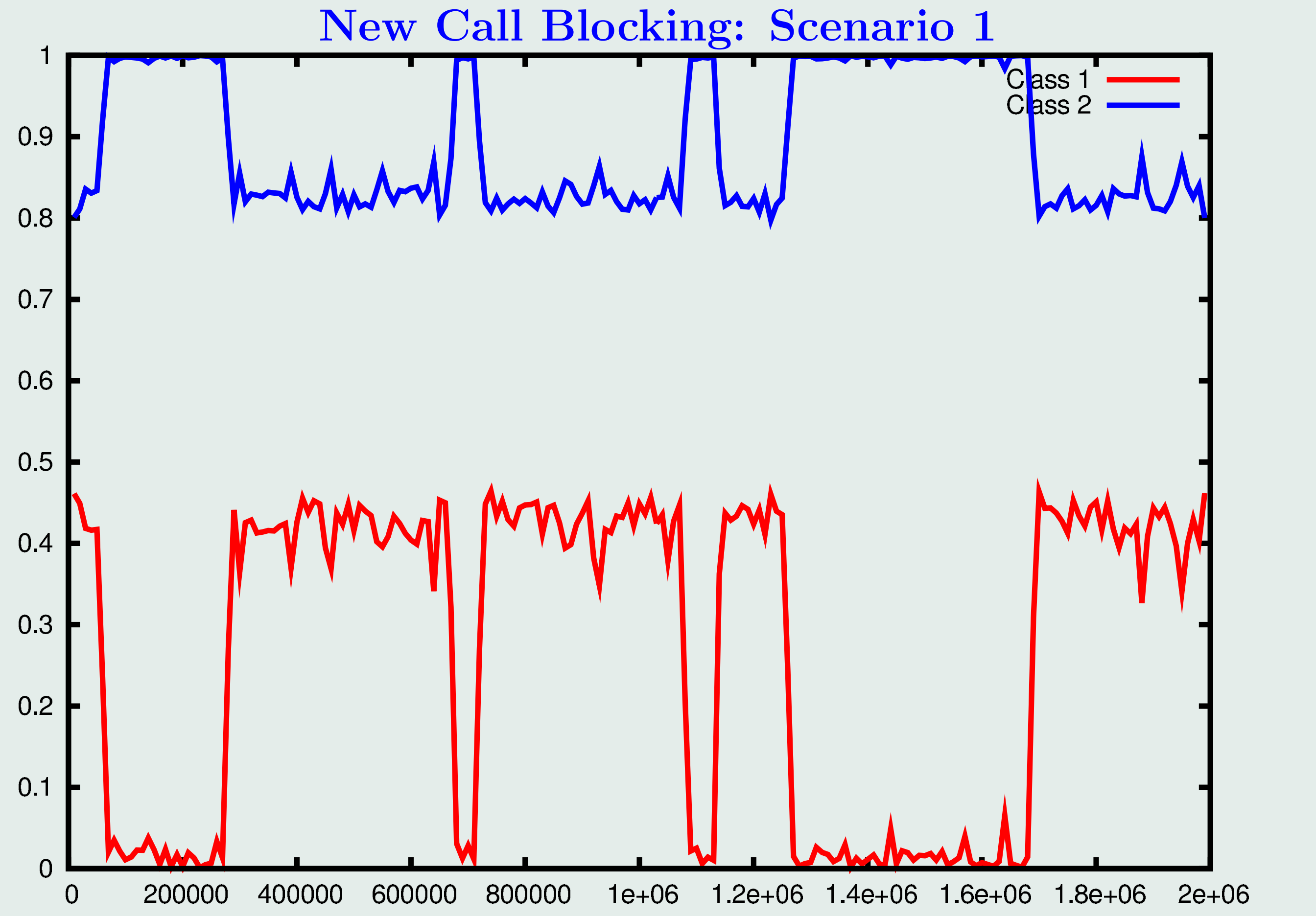
A Simulation

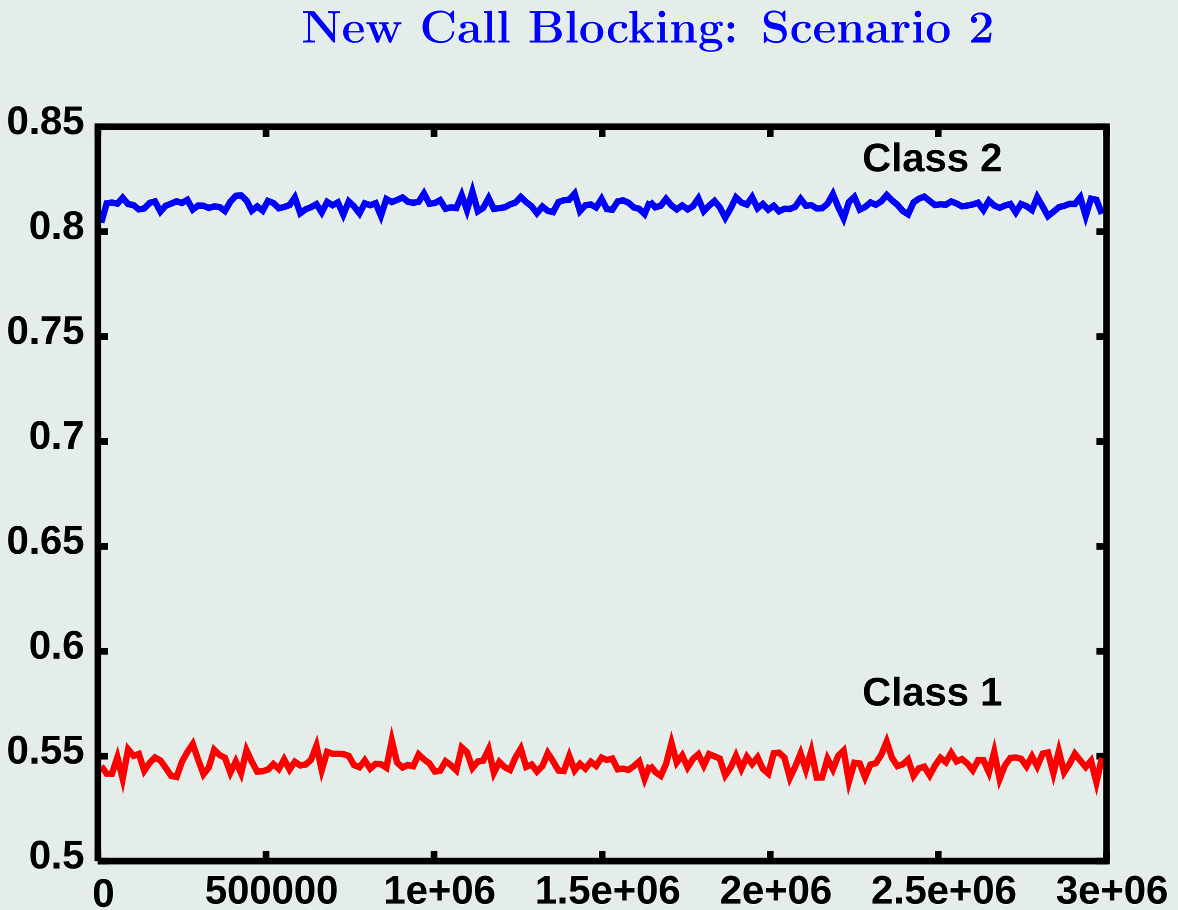
— **Two Classes of connections:**
Class 1 (Low),
Class 2 (High).

— **49 cells on a torus.**

Example




New Call Blocking: Scenario 1
Class 1
Class 2
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
200000
400000
600000
800000
1e+06
1.2e+06
1.4e+06
1.6e+06
1.8e+06
2e+06


New Call Blocking: Scenario 2
Class 2
Class 1
0.85
0.8
0.75
0.7
0.65
0.6
0.55
0.5
0
500000
1e+06
1.5e+06
2e+06
2.5e+06
3e+06

Function g revisited

For $y \in \mathcal{P}(\mathcal{X})$, $\langle \mathbb{I}_k, y \rangle = \sum_{m \in \mathcal{X}} m_k y_m$

$$g(y) = \sum_{n \in \mathcal{X}} y_n \log(n! y_n) - \sum_{k=1}^{K} \int_0^{\langle \mathbb{I}_k, y \rangle} \log \frac{\lambda_k + \gamma_k x}{\mu_k + \gamma_k} \, dx.$$

Function g revisited

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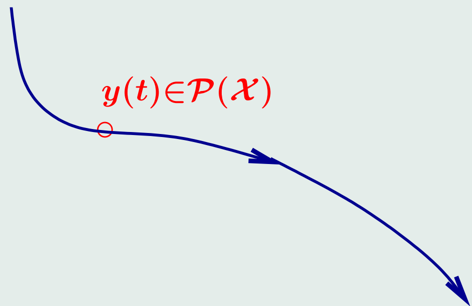
In fact, if $\rho(y) = (\rho_k(y)) = (\lambda_k + \gamma_k \langle \mathbb{I}_k, y \rangle / (\mu_k + \gamma_k))$, then

$$g(y) = C + H(y \mid \nu_{\rho(y)}) + \phi(\rho(y))$$

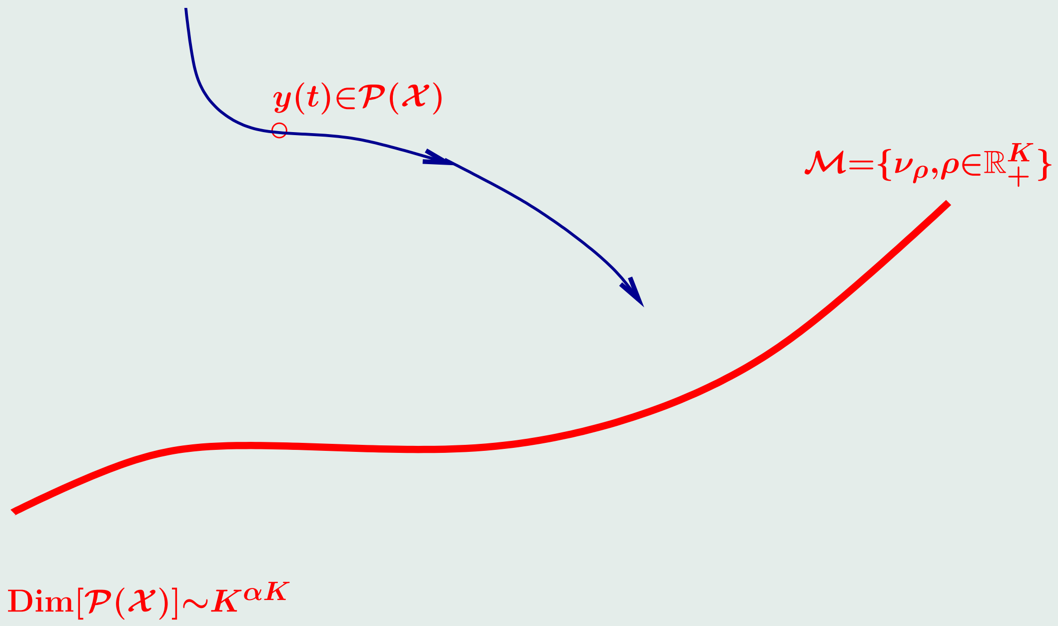
$H(\cdot \mid \cdot)$ relative entropy.

A Classical Setting

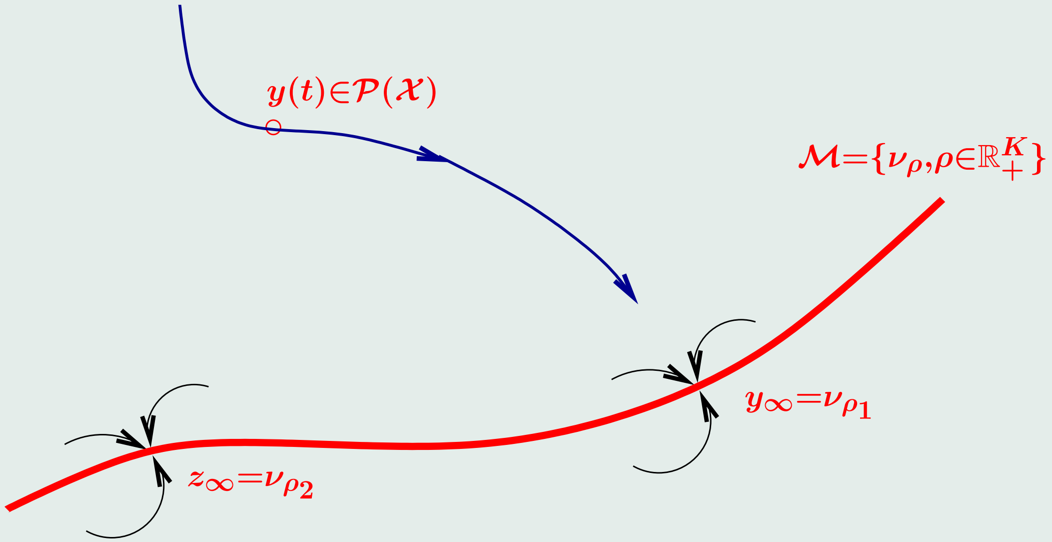


$$\mathrm{Dim}[\mathcal{P}(\mathcal{X})] \sim K^{\alpha K}$$

A Classical Setting

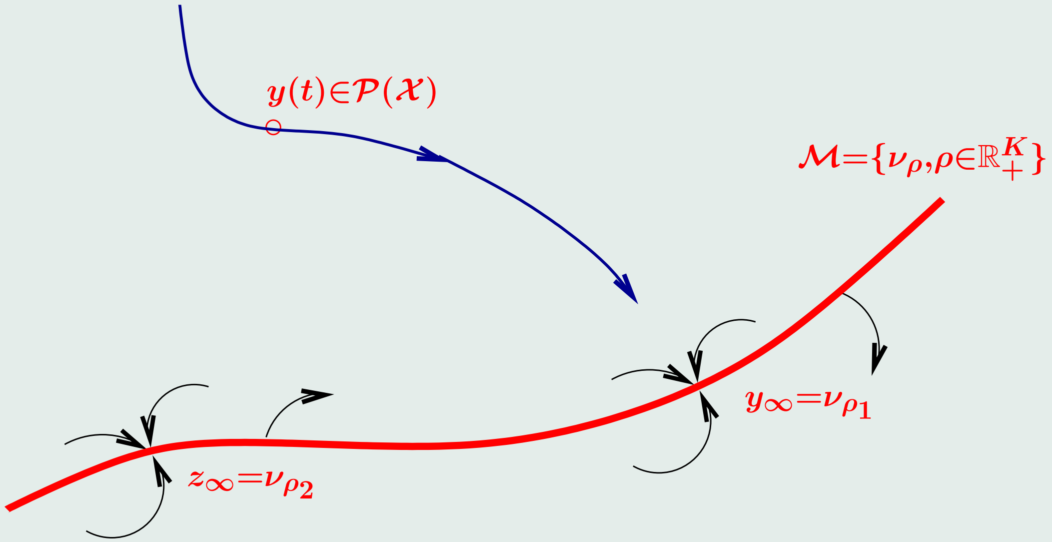


A Classical Setting



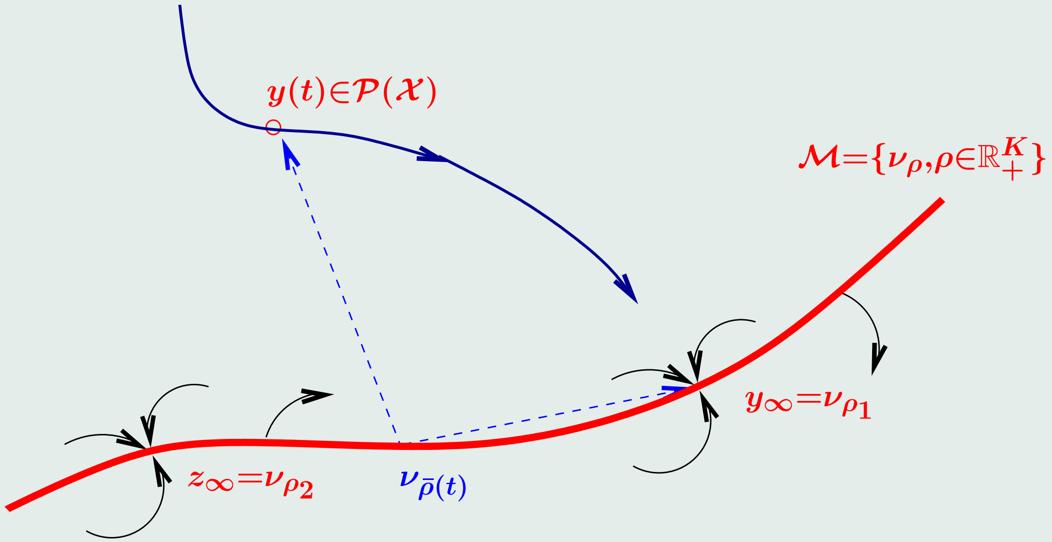
y_∞, z_∞ **Equilibrium points**

A Classical Setting



$\mathcal{M}$ NOT invariant by $(y(t))$

A Classical Setting



Relative Entropy Methods to Control Distance to $\mathcal{M}$?

Further Questions

— **With multiple stable points**
metastability property holds ?
i.e. Time to leave "stable region" $\sim \exp(\alpha N)$

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— **Relative entropy methods**
for a large class of networks ?

The end