Optimal execution strategies in limit order books

Aurélien Alfonsi

CERMICS, Mathfi Project, Ecole des Ponts, Marne-la-vallée.

Joint work with Antje Fruth (QP Lab, TU Berlin) and Alexander Schied (Cornell)

Work available at: http://cermics.enpc.fr/~alfonsi

Journées MAS, August 28th 2008, Rennes

Structure of the talk

What is a limit order book ?

The limit order book of an asset gathers all the buy and sell orders.

- Orders are made on a common discretized price grid.
- Transactions are made as soon as they can.
- Highest waiting buy order : *bid* price. Lowest waiting sell order : *ask* price. Asset value $mid = \frac{bid+ask}{2}$.
- When a possible buy (resp. sell) order is made, it is executed with the cheapest waiting sell orders (resp. most expensive waiting buy orders) according to a FIFO rule for orders at the same price.
- Different transaction costs may be applied between the waiting orders and the orders that are immediately executed.

How to take into account this when modeling ?

Limit order book are very complex objects that are rather difficult to model in their wholeness. The liquidity risk it implies is mainly treated according two points of view :

- Hedging derivatives and portfolio managment : how does liquidity risk impact the hedging strategies ? What is the extra cost it induces ? Many works on extensions to usual asset models.
- Order Execution : Once an order is made (amount and deadline), what is the optimal way to execute it ? Standard approach : statistical studies and **LOB modeling**.

The problem addressed in this talk

- Given a large number of shares X_0 and a deadline T, we want to find an **optimal buy/sell strategy** $\xi_0, \dots, \xi_N$ executed on a time-grid $t_0 < t_1 < \cdots < t_n \leq T$ such that $\sum_{n=0}^{N} \xi_n = X_0$, and **that minimizes the whole expected transaction cost**.
- We propose here a rather simple LOB model, with an intuitive parametrization.
- We will get explicit formulas for the optimal strategies. However, our modeling remains too simple to grasp the whole complexity of a real LOB dynamics.

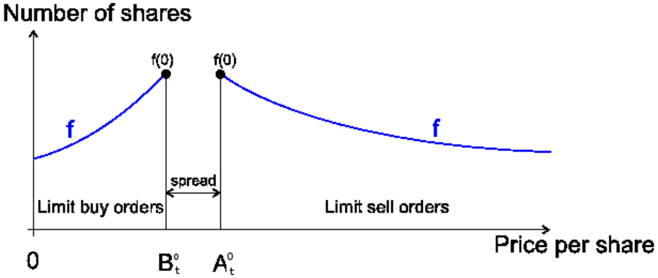
Model assumptions

- We assume that there is one large trader that aims to buy X_0 shares.
- When the large trader is not active, we assume that the ask (resp. bid) price is given by $(A^0_t, t \geq 0)$ (resp. $(B^0_t, t \geq 0)$).
- We assume that $(A^0_t, t \geq 0)$ is a martingale and $(B^0_t, t \geq 0)$ is such that $\forall t \geq 0, B^0_t \leq A^0_t$ $a.s.$ (mg assumption on B^0_t if we consider a sell order).
- The LOB is modelled as follows : the number of sell orders between prices $A^0_t + x$ and $A^0_t + x + dx$ ($x \geq 0$) is given by :

$$f(x)dx,$$

and the number of buy orders between $B^0_t + x$ and $B^0_t + x + dx$ ($x < 0$) is also $f(x)dx$. The function $f : \mathbb{R} \to \mathbb{R}^*_+$ is called the *shape function* of the LOB and is assumed to be continuous.

The LOB at time t without any trade from the large trader :



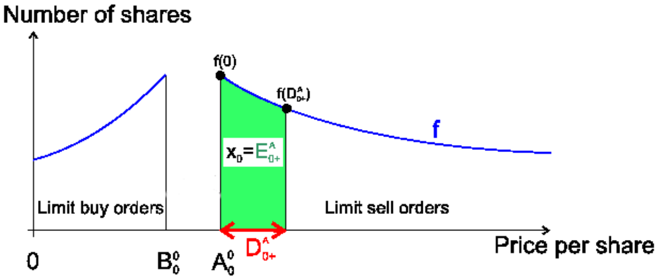
Model for large buy/sell order

Suppose at time 0 that the large trader wants to buy $x_0 > 0$ shares.

- He will consume the cheapest one between A_0^0 and $A_0^0 + D_{0+}^A$ where $\int_0^{D_{0+}^A} f(x)dx = x_0$.
- The ask price is shifted from $A_0 = A_0^0$ to $A_{0+} = A_0 + D_{0+}^A$.
- The cost of the transaction is equal to :

$$\int_0^{D_{0+}^A} (x + A^0) f(x)dx = A_0 x_0 + \int_0^{D_{0+}^A} x f(x)dx.$$

- Similarly, a sell order of $-x_0 > 0$ shares moves the bid price from $B_0 = B_0^0$ to $B_{0+} = B_0 + D_{0+}^B$ where $\int_0^{D_{0+}^B} f(x)dx = x_0$.



$E_{0+} = \int_0^{D^A_{0+}} f(x)dx$: number of shares eaten in the LOB just after the buy order.

LOB dynamics without large trade

We denote $F(x) = \int_0^x f(u)du$ and at time t :

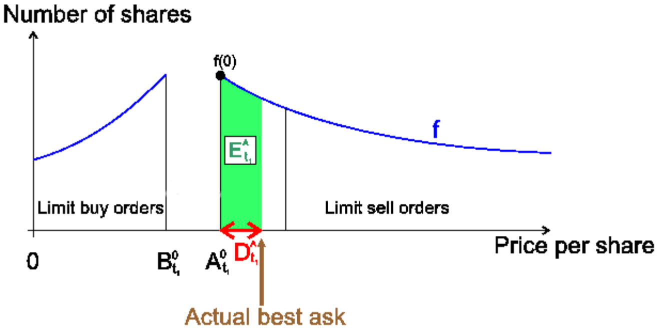
- $D_t^A := A_t - A_t^0$ (resp. $D_t^B := B_t - B_t^0$) the *extra spread* caused by the actions of the large trader.
- $E_t^A = F(D_t^A)$ (resp. $E_t^B = F(D_t^B)$) the number of sell (resp. - up to the sign - buy) orders already eaten up at time t.

If the large investor is inactive on $[t, t+s[$, we assume :

- **Model 1 :** $E_{t+s}^A = e^{-\rho s} E_t^A$ (resp. $E_{t+s}^B = e^{-\rho s} E_t^B$).
- **Model 2 :** $D_{t+s}^A = e^{-\rho s} D_t^A$ (resp. $D_{t+s}^B = e^{-\rho s} D_t^B$).

$\rho > 0$ is called the *resilience speed* of the LOB.
Rem : for $f(x) = q$ both models coincide (Obizhaeva and Wang model).

Example with no trade on $]0, t_1]$:



The cost minimization problem

At time t, a buy market order $x_t \geq 0$ moves D^A_t to D^A_{t+} s.t. $\int_{D^A_t}^{D^A_{t+}} f(x)\,dx = x_t$ and the cost is :

$$\pi_t(x_t) := \int_{D^A_t}^{D^A_{t+}} (A^0_t + x) f(x)\,dx = A^0_t x_t + \int_{D^A_t}^{D^A_{t+}} x f(x)\,dx.$$

Similarly, the cost of a sell order $x_t \leq 0$ is $\pi_t(x_t) := B^0_t x_t + \int_{D^B_t}^{D^B_{t+}} x f(x)\,dx$. Trades are allowed on the regular time grid : $t_n = n\tau$ for $n = 0, \dots, N$ and $\tau := T/N$, and an admissible strategy is s.t. $\sum_{n=0}^N \xi_n = X_0$, each ξ_n is $\mathcal{F}_{t_n}$-measurable bounded from below. The *average cost* $\mathcal{C}(\xi)$ to minimize is :

$$\boxed{\mathcal{C}(\xi) = \mathbb{E}\Big[\sum_{n=0}^N \pi_{t_n}(\xi_n)\Big].}$$

The main result for model 1

We assume from now $\lim_{x\uparrow\infty} F(x) = \infty$ and $\lim_{x\downarrow-\infty} F(x) = -\infty$. Suppose $h_1(u) := F^{-1}(u) - e^{-\rho\tau} F^{-1}(e^{-\rho\tau} u)$ one-to-one. Then there exists a unique optimal strategy $\xi^{(1)} = (\xi_0^{(1)}, \dots, \xi_N^{(1)})$.

$\xi_0^{(1)}$: unique solution of the equation

$$F^{-1}\left(X_0 - N\xi_0^{(1)}\left(1 - e^{-\rho\tau}\right)\right) = \frac{h_1(\xi_0^{(1)})}{1 - e^{-\rho\tau}},$$

the intermediate orders are given by

$$\xi_1^{(1)} = \cdots = \xi_{N-1}^{(1)} = \xi_0^{(1)}\left(1 - e^{-\rho\tau}\right),$$

the final order is determined by

$$\xi_N^{(1)} = X_0 - \xi_0^{(1)} - (N-1)\xi_0^{(1)}\left(1 - e^{-\rho\tau}\right).$$

It is deterministic and s.t. $\xi_n^{(1)} > 0$ for all n.

The main result for model 2

Suppose $h_2(x) := x\frac{f(x)-e^{-2\rho\tau}f(e^{-\rho\tau}x)}{f(x)-e^{-\rho\tau}f(e^{-\rho\tau}x)}$ one-to-one, and $\lim_{|x|\to\infty} x^2 \inf_{y\in[e^{-\rho\tau}x,x]} f(y) = \infty$. Then there exists a unique optimal strategy $\xi^{(2)} = (\xi_0^{(2)}, \dots, \xi_N^{(2)})$.

$\xi_0^{(2)}$: unique solution of the equation

$$F^{-1}\left(X_0 - N\left[\xi_0^{(2)} - F\left(e^{-\rho\tau}F^{-1}(\xi_0^{(2)})\right)\right]\right) = h_2\left(F^{-1}(\xi_0^{(2)})\right),$$

the intermediate orders are given by

$$\xi_1^{(2)} = \cdots = \xi_{N-1}^{(2)} = \xi_0^{(2)} - F\left(e^{-\rho\tau}F^{-1}(\xi_0^{(2)})\right)$$

the final order is determined by

$$\xi_N^{(2)} = X_0 - N\xi_0^{(2)} + (N-1)F\left(e^{-\rho\tau}F^{-1}(\xi_0^{(2)})\right).$$

It is deterministic and s.t. $\xi_n^{(2)} > 0$ for all n.

Comments

- Optimal strategies have a clear interpretation in both models : the first trade shifts the ask price to the best trade-off between price and attracting new orders.
- One can show that h_1 is one-to-one if f is increasing on $\mathbb{R}_-$ and decreasing on $\mathbb{R}_+$. There is no such simple characterization for h_2.
- In the case $f(x) = q$ (block-shaped LOB), both theorems give the following optimal strategy :

$$\xi_0^* = \xi_N^* = \frac{X_0}{(N-1)(1-e^{-\rho\tau})+2} \quad \text{and} \quad \xi_1^* = \cdots = \xi_{N-1}^* = \frac{X_0 - 2\xi_0^*}{N-1}.$$

It does not depend on q.

Examples of shape functions

Example number	$f(x)$	Model 1 $\xi_0^{(1)}$	Model 1 $\xi_1^{(1)}$	Model 1 $\xi_N^{(1)}$	Model 2 $\xi_0^{(2)}$	Model 2 $\xi_1^{(2)}$	Model 2 $\xi_N^{(2)}$
0	q	10,223	8,839	10,223	10,223	8,839	10,223
1	$\frac{q}{\sqrt{\lvert x\rvert+1}}$	10,257	8,869	9,925	10,756	8,724	10,726
2	$\frac{q}{\lvert x\rvert+1}$	10,303	8,909	9,520	13,305	8,154	13,305
3	$qe^{\lvert x\rvert}$	10,139	8,767	10,962	9,735	8,947	9,741
4	$\frac{q}{10}\lvert x\rvert + q$	10,211	8,829	10,326	10,130	8,860	10,131
5	$\frac{q}{10}x^2 + q$	10,192	8,812	10,498	10,101	8,868	10,091

TAB.: $X_0 = 100,000$, $q = 5,000$, $\rho = 20$, $T = 1$ and $N = 10$.

Sketch of the proof

- Thanks to the martingale assumption on A_t^0, it is sufficient to minimize in both models $i \in \{1, 2\}$
$$\mathbb{E}[C^{(i)}(\xi_0, \dots, \xi_N)]$$
where $C^{(i)}$ is a deterministic function.
- It is then sufficient to show that $C^{(i)}$ has a unique minimizer in $\Xi = \left\{ (x_0, \dots, x_N) \in \mathbb{R}^{N+1} \mid \sum_{n=0}^{N} x_n = X_0 \right\}$, which can be done using a Lagrange multiplier.

Model description

- The unaffected price A_t^0 follows a Bachelier model, but this could be as previously a martingale.
- They assume a block-shaped LOB ($f(x) = q$), but allow a *permanent impact* on the extra spread D_t^A :

$$D_t^A = \gamma \sum_{t_k<t} \xi_k + \sum_{t_k<t} \kappa e^{-\rho(t-t_k)} \xi_k, \tag{1}$$

  where $\kappa := \frac{1}{q} - \gamma$ and $\gamma < 1/q$ is a constant quantifying the permanent impact.
- For $\gamma = 0$, it is a particular case of the previous model.
- Trades occur on the regular grid : $t_n = n\tau, n = 0, \dots, N$.
- Using the dynamic programming principle, they get the following result.

Obizhaeva and Wang main result

In a block-shaped LOB with permanent impact γ, the optimal strategy ξ^{OW} in the class of deterministic strategies is determined by the scheme

$$\begin{aligned} \xi_n^{OW} &= \frac{1}{2}\delta_{n+1}\left[\epsilon_{n+1}X_{t_n} - \phi_{n+1}D_{t_n}\right], \qquad n = 0,\dots,N-1, \\ \xi_N^{OW} &= X_T, \end{aligned}$$

where δ_n, ϵ_n and ϕ_n are defined by the backward scheme

$$\begin{aligned} \delta_n &:= \left(\frac{1}{2q} + \alpha_n - \beta_n \kappa e^{-\rho\tau} + \gamma_n \kappa^2 e^{-2\rho\tau}\right)^{-1} \\ \epsilon_n &:= \gamma + 2\alpha_n - \beta_n \kappa e^{-\rho\tau} \\ \phi_n &:= 1 - \beta_n e^{-\rho\tau} + 2\gamma_n \kappa e^{-2\rho\tau}. \end{aligned}$$

with α_n, β_n and γ_n given by

$$\begin{aligned} \alpha_N = \frac{1}{2q} - \gamma \quad \text{and} \quad \alpha_n &= \alpha_{n+1} - \frac{1}{4}\delta_{n+1}\epsilon_{n+1}^2, \\ \beta_N = 1 \quad \text{and} \quad \beta_n &= \beta_{n+1}e^{-\rho\tau} + \frac{1}{2}\delta_{n+1}\epsilon_{n+1}\phi_{n+1}, \\ \gamma_N = 0 \quad \text{and} \quad \gamma_n &= \gamma_{n+1}e^{-2\rho\tau} - \frac{1}{4}\delta_{n+1}\phi_{n+1}^2. \end{aligned}$$

An explicit solution

As before, the minimization problem amounts to minimize a deterministic function given by : $C^{\mathrm{OW}}_{\gamma,q}(x_0,\dots,x_N) =$
$A_0 \sum_{i=0}^N x_i + \frac{\gamma}{2}\left(\sum_{i=0}^N x_i\right)^2 + \kappa \sum_{k=0}^N \left(\sum_{i=0}^{k-1} x_i e^{-\rho(k-i)\tau}\right) x_k + \frac{\kappa}{2}\sum_{i=0}^N x_i^2.$
With this writing, we see that

$$C^{\mathrm{OW}}_{\gamma,q}(x_0,\dots,x_N) = \frac{\gamma}{2}\left(\sum_{i=0}^N x_i\right)^2 + C^{\mathrm{OW}}_{0,\kappa^{-1}}(x_0,\dots,x_N)$$

and under the constraint $\sum_{i=0}^N x_i = X_0$, it is equivalent to minimize either $C^{\mathrm{OW}}_{\gamma,q}$ or $C^{\mathrm{OW}}_{0,\kappa^{-1}}$.
$\Longrightarrow$ The optimal strategy is then given by
$\xi_0^* = \xi_N^* = \frac{X_0}{(N-1)(1-e^{-\rho\tau})+2}$ and $\xi_1^* = \dots = \xi_{N-1}^* = \frac{X_0 - 2\xi_0^*}{N-1}$.

Extension to non-regular time-grids and time-dependent resilience

$t_0 = 0 < t_1 < \cdots < t_N = T$. Denote $a_0 = 0$ and $a_n = e^{-\int_{t_{n-1}}^{t_n} \rho_s \, ds}$.
There exists a unique optimal strategy $\xi^* = (\xi_0^*, \ldots, \xi_N^*)$ in the class of all admissible strategies. With the notation $\lambda_0 := \frac{X_0}{\frac{2}{1+a_1} + \sum_{n=2}^N \frac{1-a_n}{1+a_n}}$, the initial market order is

$$\xi_0^* = \frac{\lambda_0}{1+a_1},$$

the intermediate market orders are given by

$$\xi_n^* = \lambda_0 \Big(\frac{1}{1+a_n} - \frac{a_{n+1}}{1+a_{n+1}} \Big), \qquad n = 1, \ldots, N-1,$$

and the final market order is

$$\xi_N^* = \frac{\lambda_0}{1+a_N}.$$

It is deterministic and such that $\xi_n^* > 0$ for all n.
NB : It is easy to add linear constraints using Kuhn-Tucker Theorem.

Sum up

We have extended and improved the Obizhaeva and Wang model in the following directions :

- explicit optimal solutions
- general LOB shape functions
- constrained optimization for time dependent resilience speed and non-regular time-grid.

These are encouraging results.

Possible improvements to be closer to real LOB

- Many large traders instead of one (add stochastic jumps to the processes D^A and D^B).
- In our setting, we only buy sell orders and do not consider the possibility of putting waiting buy orders in the LOB.
- (stochastic) time-dependent shape function.
- ...

But of course, at the end, one should have a trade-off between realism and tractability.