Optimal execution strategies in limit order books

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What is a limit order book?

The limit order book of an asset gathers all the buy and sell orders.

- Orders are made on a common discretized price grid.
- Transactions are made as soon as they can.
- Highest waiting buy order: *bid* price. Lowest waiting sell order: *ask* price. Asset value $\text{mid} = \frac{\text{bid} + \text{ask}}{2}$.
- When a possible buy (resp. sell) order is made, it is executed with the cheapest waiting sell orders (resp. most expensive waiting buy orders) according to a FIFO rule for orders at the same price.
- Different transaction costs may be applied between the waiting orders and the orders that are immediately executed.
How to take into account this when modeling?

Limit order book are very complex objects that are rather difficult to model in their wholeness. The liquidity risk it implies is mainly treated according two points of view:

- **Hedging derivatives and portfolio management**: how does liquidity risk impact the hedging strategies? What is the extra cost it induces? Many works on extensions to usual asset models.

- **Order Execution**: Once an order is made (amount and deadline), what is the optimal way to execute it? Standard approach: statistical studies and **LOB modeling**.
The problem addressed in this talk

- Given a large number of shares $X_0$ and a deadline $T$, we want to find an optimal buy/sell strategy $\xi_0, \ldots, \xi_N$ executed on a time-grid $t_0 < t_1 < \cdots < t_n \leq T$ such that $\sum_{n=0}^{N} \xi_n = X_0$, and that minimizes the whole expected transaction cost.

- We propose here a rather simple LOB model, with an intuitive parametrization.

- We will get explicit formulas for the optimal strategies. However, our modeling remains too simple to grasp the whole complexity of a real LOB dynamics.
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Model assumptions

- We assume that there is one large trader that aims to buy $X_0$ shares.
- When the large trader is not active, we assume that the ask (resp. bid) price is given by $(A^0_t, t \geq 0)$ (resp. $(B^0_t, t \geq 0)$).
- We assume that $(A^0_t, t \geq 0)$ is a martingale and $(B^0_t, t \geq 0)$ is such that $\forall t \geq 0, B^0_t \leq A^0_t$ a.s. (mg assumption on $B^0_t$ if we consider a sell order).
- The LOB is modelled as follows: the number of sell orders between prices $A^0_t + x$ and $A^0_t + x + dx$ ($x \geq 0$) is given by:
  \[ f(x)dx, \]
  and the number of buy orders between $B^0_t + x$ and $B^0_t + x + dx$ ($x < 0$) is also $f(x)dx$. The function $f : \mathbb{R} \to \mathbb{R}_+^*$ is called the shape function of the LOB and is assumed to be continuous.
The LOB at time $t$ without any trade from the large trader:

Number of shares

Limit buy orders

Limit sell orders

Price per share

$B_t^0$ $A_t^0$
Model for large buy/sell order

Suppose at time 0 that the large trader wants to buy $x_0 > 0$ shares.

- He will consume the cheapest one between $A_0^0$ and $A_0^0 + D_{0+}^A$
  where $\int_0^{D_{0+}^A} f(x)dx = x_0$.
- The ask price is shifted from $A_0 = A_0^0$ to $A_{0+} = A_0 + D_{0+}^A$.
- The cost of the transaction is equal to:
  \[ \int_0^{D_{0+}^A} (x + A_0^0)f(x)dx = A_0x_0 + \int_0^{D_{0+}^A} xf(x)dx. \]

- Similarly, a sell order of $-x_0 > 0$ shares moves the bid price from $B_0 = B_0^0$ to $B_{0+} = B_0 + D_{0+}^B$ where $\int_0^{D_{0+}^B} f(x)dx = x_0$. 
$E_{0+} = \int_{D_{0+}'}^{D_A} f(x)dx$ : number of shares eaten in the LOB just after the buy order.
LOB dynamics without large trade

We denote $F(x) = \int_0^x f(u)du$ and at time $t$:

- $D^A_t := A_t - A_0^t$ (resp. $D^B_t := B_t - B_0^t$) the extra spread caused by the actions of the large trader.
- $E^A_t = F(D^A_t)$ (resp. $E^B_t = F(D^B_t)$) the number of sell (resp. - up to the sign - buy) orders already eaten up at time $t$.

If the large investor is inactive on $[t, t+s]$, we assume:

- **Model 1**: $E^A_{t+s} = e^{-\rho s} E^A_t$ (resp. $E^B_{t+s} = e^{-\rho s} E^B_t$).
- **Model 2**: $D^A_{t+s} = e^{-\rho s} D^A_t$ (resp. $D^B_{t+s} = e^{-\rho s} D^B_t$).

$\rho > 0$ is called the resilience speed of the LOB.

**Rem**: for $f(x) = q$ both models coincide (Obizhaeva and Wang model).
Example with no trade on $[0, t_1]$:

![Diagram showing the number of shares and price per share with limit buy and sell orders.](image-url)
The cost minimization problem

At time $t$, a buy market order $x_t \geq 0$ moves $D_t^A$ to $D_{t+}^A$ s.t.

$$\int_{D_t^A}^{D_{t+}^A} f(x) \, dx = x_t$$

and the cost is:

$$\pi_t(x_t) := \int_{D_t^A}^{D_{t+}^A} (A_t^0 + x)f(x) \, dx = A_t^0 x_t + \int_{D_t^A}^{D_{t+}^A} xf(x) \, dx.$$  

Similarly, the cost of a sell order $x_t \leq 0$ is $\pi_t(x_t) := B_t^0 x_t + \int_{D_t^B}^{D_{t+}^B} xf(x) \, dx$.

Trades are allowed on the regular time grid: $t_n = n\tau$ for $n = 0, \ldots, N$ and $\tau := T/N$, and an admissible strategy is s.t. $\sum_{n=0}^{N} \xi_n = X_0$, each $\xi_n$ is $\mathcal{F}_{t_n}$-measurable bounded from below. The average cost $C(\xi)$ to minimize is:

$$C(\xi) = \mathbb{E}\left[ \sum_{n=0}^{N} \pi_t(x_{\xi_n}) \right].$$
The main result for model 1

We assume from now \( \lim_{x \uparrow \infty} F(x) = \infty \) and \( \lim_{x \downarrow -\infty} F(x) = -\infty \).

Suppose \( h_1(u) := F^{-1}(u) - e^{-\rho \tau} F^{-1}(e^{-\rho \tau} u) \) one-to-one. Then there exists a unique optimal strategy \( \xi^{(1)} = (\xi_0^{(1)}, \ldots, \xi_N^{(1)}) \).

\( \xi_0^{(1)} \) : unique solution of the equation

\[
F^{-1} \left( X_0 - N \xi_0^{(1)} \left( 1 - e^{-\rho \tau} \right) \right) = \frac{h_1(\xi_0^{(1)})}{1 - e^{-\rho \tau}},
\]

the intermediate orders are given by

\[
\xi_1^{(1)} = \cdots = \xi_{N-1}^{(1)} = \xi_0^{(1)} \left( 1 - e^{-\rho \tau} \right),
\]

the final order is determined by

\[
\xi_N^{(1)} = X_0 - \xi_0^{(1)} - (N - 1) \xi_0^{(1)} \left( 1 - e^{-\rho \tau} \right).
\]

It is deterministic and s.t. \( \xi_n^{(1)} > 0 \) for all \( n \).
The main result for model 2

Suppose $h_2(x) := x\frac{f(x) - e^{2\rho \tau} f(e^{-\rho \tau} x)}{f(x) - e^{-\rho \tau} f(e^{-\rho \tau} x)}$ one-to-one, and

$$\lim_{|x| \to \infty} x^2 \inf_{y \in [e^{-\rho x}, x]} f(y) = \infty.$$

Then there exists a unique optimal strategy $\xi^{(2)} = (\xi_0^{(2)}, \ldots, \xi_N^{(2)})$. 

$\xi_0^{(2)}$ : unique solution of the equation

$$F^{-1} \left( X_0 - N \left[ \xi_0^{(2)} - F(e^{-\rho \tau} F^{-1}(\xi_0^{(2)})) \right] \right) = h_2(F^{-1}(\xi_0^{(2)})), $$

the intermediate orders are given by

$$\xi_1^{(2)} = \cdots = \xi_{N-1}^{(2)} = \xi_0^{(2)} - F(e^{-\rho \tau} F^{-1}(\xi_0^{(2)}))$$

the final order is determined by

$$\xi_N^{(2)} = X_0 - N \xi_0^{(2)} + (N - 1)F(e^{-\rho \tau} F^{-1}(\xi_0^{(2)})).$$

It is deterministic and s.t. $\xi_n^{(2)} > 0$ for all $n$. 
Comments

- Optimal strategies have a clear interpretation in both models: the first trade shifts the ask price to the best trade-off between price and attracting new orders.
- One can show that \( h_1 \) is one-to-one if \( f \) is increasing on \( \mathbb{R}_- \) and decreasing on \( \mathbb{R}_+ \). There is no such simple characterization for \( h_2 \).
- In the case \( f(x) = q \) (block-shaped LOB), both theorems give the following optimal strategy:

\[
\xi_0^* = \xi_N^* = \frac{X_0}{(N - 1)(1 - e^{-\rho \tau}) + 2} \quad \text{and} \quad \xi_1^* = \cdots = \xi_{N-1}^* = \frac{X_0 - 2\xi_0^*}{N - 1}.
\]

It does not depend on \( q \).
## Examples of shape functions

<table>
<thead>
<tr>
<th>Example number</th>
<th>$f(x)$</th>
<th>Model 1</th>
<th>Model 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$q$</td>
<td>$\xi_0^{(1)}$</td>
<td>$\xi_1^{(1)}$</td>
</tr>
<tr>
<td>1</td>
<td>$\frac{q}{\sqrt{</td>
<td>x</td>
<td>+1}}$</td>
</tr>
<tr>
<td>2</td>
<td>$\frac{q}{</td>
<td>x</td>
<td>+1}$</td>
</tr>
<tr>
<td>3</td>
<td>$qe^{</td>
<td>x</td>
<td>}$</td>
</tr>
<tr>
<td>4</td>
<td>$\frac{q}{10}</td>
<td>x</td>
<td>+ q$</td>
</tr>
<tr>
<td>5</td>
<td>$\frac{q}{10}x^2 + q$</td>
<td>10,192</td>
<td>8,812</td>
</tr>
</tbody>
</table>

**Tab.:** $X_0 = 100,000$, $q = 5,000$, $\rho = 20$, $T = 1$ and $N = 10$. 
Sketch of the proof

- Thanks to the martingale assumption on $A^0_t$, it is sufficient to minimize in both models $i \in \{1, 2\}$

$$\mathbb{E}[C^{(i)}(\xi_0, \ldots, \xi_N)]$$

where $C^{(i)}$ is a deterministic function.

- It is then sufficient to show that $C^{(i)}$ has a unique minimizer in $\Xi = \left\{ (x_0, \ldots, x_N) \in \mathbb{R}^{N+1} \mid \sum_{n=0}^{N} x_n = X_0 \right\}$, which can be done using a Lagrange multiplier.
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Model description

- The unaffected price $A_t^0$ follows a Bachelier model, but this could be as previously a martingale.
- They assume a block-shaped LOB ($f(x) = q$), but allow a permanent impact on the extra spread $D_t^A$:

$$D_t^A = \gamma \sum_{t_k < t} \xi_k + \sum_{t_k < t} \kappa e^{-\rho(t-t_k)} \xi_k,$$

where $\kappa := \frac{1}{q} - \gamma$ and $\gamma < 1/q$ is a constant quantifying the permanent impact.
- For $\gamma = 0$, it is a particular case of the previous model.
- Trades occur on the regular grid: $t_n = n\tau$, $n = 0, \ldots, N$.
- Using the dynamic programming principle, they get the following result.
Obizhaeva and Wang main result

In a block-shaped LOB with permanent impact $\gamma$, the optimal strategy $\xi^{OW}$ in the class of deterministic strategies is determined by the scheme

$$
\xi^{OW}_n = \frac{1}{2} \delta_{n+1} [\epsilon_{n+1} X_{t_n} - \phi_{n+1} D_{t_n}] , \quad n = 0, \ldots, N - 1,
$$

$$
\xi^{OW}_N = X_T,
$$

where $\delta_n$, $\epsilon_n$ and $\phi_n$ are defined by the backward scheme

$$
\delta_n := \left( \frac{1}{2q} + \alpha_n - \beta_n \kappa e^{-\rho \tau} + \gamma_n \kappa^2 e^{-2\rho \tau} \right)^{-1}
$$

$$
\epsilon_n := \gamma + 2\alpha_n - \beta_n \kappa e^{-\rho \tau}
$$

$$
\phi_n := 1 - \beta_n e^{-\rho \tau} + 2\gamma_n \kappa e^{-2\rho \tau}.
$$

with $\alpha_n$, $\beta_n$ and $\gamma_n$ given by

$$
\alpha_N = \frac{1}{2q} - \gamma \quad \text{and} \quad \alpha_n = \alpha_{n+1} - \frac{1}{4} \delta_{n+1} \epsilon_{n+1}^2,
$$

$$
\beta_N = 1 \quad \text{and} \quad \beta_n = \beta_{n+1} e^{-\rho \tau} + \frac{1}{2} \delta_{n+1} \epsilon_{n+1} \phi_{n+1},
$$

$$
\gamma_N = 0 \quad \text{and} \quad \gamma_n = \gamma_{n+1} e^{-2\rho \tau} - \frac{1}{4} \delta_{n+1} \phi_{n+1}^2.
$$
An explicit solution

As before, the minimization problem amounts to minimize a deterministic function given by:

\[ C_{\gamma, q}^{OW}(x_0, \ldots, x_N) = A_0 \sum_{i=0}^{N} x_i + \frac{\gamma}{2} \left( \sum_{i=0}^{N} x_i \right)^2 + \kappa \sum_{k=0}^{N} \left( \sum_{i=0}^{k-1} x_i e^{-\rho(k-i)\tau} \right) x_k + \frac{\kappa}{2} \sum_{i=0}^{N} x_i^2. \]

With this writing, we see that

\[ C_{\gamma, q}^{OW}(x_0, \ldots, x_N) = \frac{\gamma}{2} \left( \sum_{i=0}^{N} x_i \right)^2 + C_{0, \kappa-1}^{OW}(x_0, \ldots, x_N) \]

and under the constraint \( \sum_{i=0}^{N} x_i = X_0 \), it is equivalent to minimize either \( C_{\gamma, q}^{OW} \) or \( C_{0, \kappa-1}^{OW} \).

\[ \Rightarrow \text{ The optimal strategy is then given by} \]

\[ \xi_0^* = \xi_N^* = \frac{X_0}{(N-1)(1-e^{-\rho\tau})+2} \quad \text{and} \quad \xi_1^* = \cdots = \xi_{N-1}^* = \frac{X_0-2\xi_0^*}{N-1}. \]
Extension to non-regular time-grids and time-dependent resilience

\[ t_0 = 0 < t_1 < \cdots < t_N = T. \]

Denote \( a_0 = 0 \) and \( a_n = e^{-\int_{t_{n-1}}^{t_n} \rho_s ds} \).

There exists a unique optimal strategy \( \xi^* = (\xi_0^*, \ldots, \xi_N^*) \) in the class of all admissible strategies. With the notation \( \lambda_0 := \frac{\frac{X_0}{2+\sum_{n=2}^{N} \frac{1}{1+a_n}}}{1+a_1 + \sum_{n=2}^{N} \frac{1-a_n}{1+a_n}} \), the initial market order is

\[ \xi_0^* = \frac{\lambda_0}{1+a_1}, \]

the intermediate market orders are given by

\[ \xi_n^* = \lambda_0 \left( \frac{1}{1+a_n} - \frac{a_{n+1}}{1+a_{n+1}} \right), \quad n = 1, \ldots, N - 1, \]

and the final market order is

\[ \xi_N^* = \frac{\lambda_0}{1+a_N}. \]

It is deterministic and such that \( \xi_n^* > 0 \) for all \( n \).

NB : It is easy to add linear constraints using Kuhn-Tucker Theorem.
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Sum up

We have extended and improved the Obizhaeva and Wang model in the following directions:

- explicit optimal solutions
- general LOB shape functions
- constrained optimization for time dependent resilience speed and non-regular time-grid.

These are encouraging results.
Possible improvements to be closer to real LOB

- Many large traders instead of one (add stochastic jumps to the processes $D^A$ and $D^B$).
- In our setting, we only buy sell orders and do not consider the possibility of putting waiting buy orders in the LOB.
- (stochastic) time-dependent shape function.
- ...

But of course, at the end, one should have a trade-off between realism and tractability.