Importance Sampling for Failure Recovery Probabilities in Computing and Data Transmission

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Efficient Monte Carlo:
From Variance Reduction to
Combinatorial Optimization

A Conference on the Occasion of

R.Y. Rubinstein's 70th Birthday

Sandbjerg, July 14, 2008

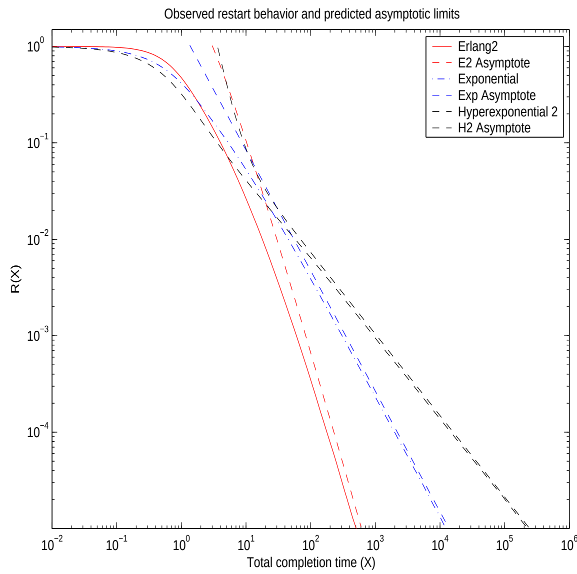
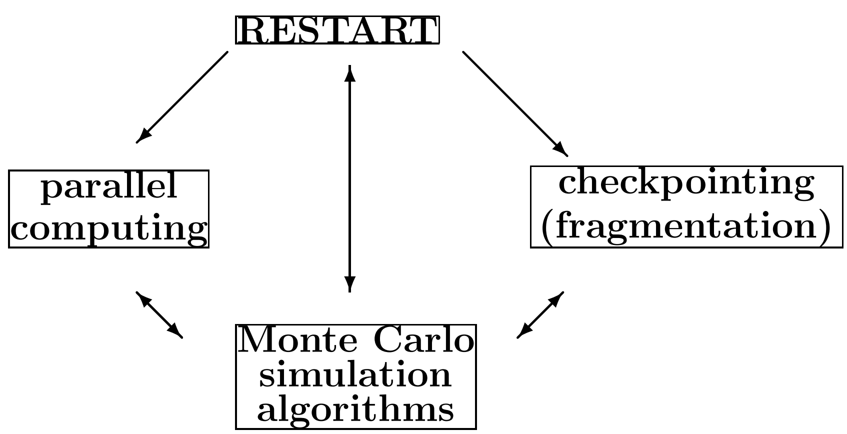


Figure 1: **Sheahan, Lipsky, Fiorini 05**

Power tails?

SA, Fiorini, Lipsky, Rolski, Sheahan
Asymptotic total task times for tasks that must restart after a failure occurs
***Mathematics of Operations Research* 08/09**

(related work of Jelenkovic & Tan)

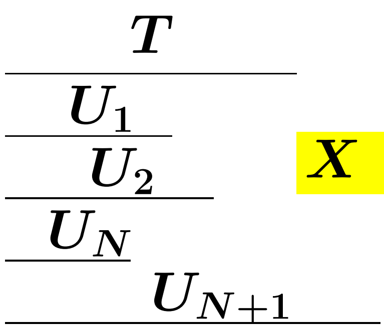


Lars Nørvang Andersen, SA+ 2-3 working papers

1. **RESTART asymptotics**
2. **IS when $T \equiv t$**
3. **IS when T gamma-like**
4. **Role of rootfinding**
5. **IS when T reg. var.**
6. **Running time issues**

RESTART

T ideal job time $\sim\ F$
(program time; file length;
call center time)
U failure time $\sim\ G$
(often exponential)
X total time $\sim\ H$



$$X = T + U_1 + \cdots + U_N$$

Target: tail $\overline{H}(x) = \mathbb{P}(X > x)$

T bounded $\Rightarrow\ \overline{H}(x) \approx \mathrm{e}^{-\gamma x}$

T unbd $\Rightarrow\ H$ heavy-tailed

4 examples of each of F, G:

LT Weibull
exponential
HT Weibull
power

. $\overline{F}(t)$ / $\overline{G}(u)$	e^{-t^2}	e^{-t}	$e^{-t^{1/2}}$	$\frac{1}{t^\alpha}$
e^{-u^2}	$\frac{1}{x}$	$e^{-\log^{1/2} x}$	$e^{-\log^{1/4} x}$	$\frac{1}{\log^{\alpha/2} x}$
e^{-u}	$e^{-\log^2 x}$	$\frac{1}{x}$	$e^{-\log^{1/2} x}$	$\frac{1}{\log^{\alpha} x}$
$e^{-u^{1/2}}$	$e^{-\log^4 x}$	$e^{-\log^2 x}$	$\frac{1}{x}$	$\frac{1}{\log^{2\alpha} x}$
$\frac{1}{u^\alpha}$	$e^{-x^{\frac{2}{2+\alpha}}}$	$e^{-x^{\frac{1}{1+\alpha}}}$	$e^{-x^{\frac{1/2}{1/2+\alpha}}}$	$\frac{1}{x}$

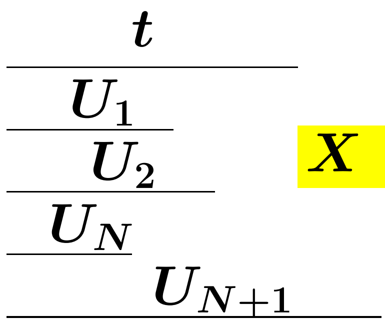
Constants omitted $e^{-c\log^{1/2} x}$; $\dfrac{1}{x} = e^{-\log x}$

In some corners even $\log\log$ **asymptotics**

Omitted case: T **bounded (later)**

Also: cases of exact asymptotics (close-to-diagonal)

RESTART with $T \equiv t$, G exponential(μ)



$$X = t + U_1 + \cdots + U_N$$

Geometric sum

Geometric Sums and Renewal Theory

$S_N = V_1 + \cdots + V_N$
$V_n \sim F$, N geom(ρ)
$Z(x) = \mathbb{P}(S_N > x)$ solves

$$Z(x) = \rho\overline{F}(x) + \int_0^x Z(x-y)\,\rho F(\mathrm{d}y) \quad (1)$$

Defective renewal equation
Solution exponentially decaying
Lundberg inequality

$\rho F(dy)$ has mass $\rho < 1$
Choose γ as solution of

$$\int_0^\infty \mathrm{e}^{\gamma x}\,\rho F(\mathrm{d}x) = 1$$

(assumes light tails)

Let $F^*(\mathrm{d}x) = \mathrm{e}^{\gamma x}\,\rho F(\mathrm{d}x)$
. . $Z^*(x) = \mathrm{e}^{\gamma x} Z(x)$

. . $z^*(x) = \mathrm{e}^{\gamma x}\rho\overline{F}(x)$

Multiply (1) by $\mathrm{e}^{\gamma x}$

$Z(x) \sim C\mathrm{e}^{-\gamma x}$

Cramér-Lundberg approximation

Geometric Sums and Change of Measure

$$\rho \int_0^\infty e^{\gamma x} F(dx) = 1$$

$$\widetilde{F}(dx) \;=\; \rho e^{\gamma x} F(dx)$$

$$\tau(x) = \inf\{n :\; S_n > x\}$$

$$\begin{aligned}\mathbb{P}(S_N > x) &= \mathbb{P}(N \geq \tau(x)) \\ &= \widetilde{\mathbb{E}}\big[\mathrm{LR}; N \geq \tau(x)\big] \\ &= \widetilde{\mathbb{E}}\big[\rho^{-\tau(x)} e^{-\gamma S_{\tau(x)}} \rho^{\tau(x)}\big] \\ &= \widetilde{\mathbb{E}} e^{-\gamma S_{\tau(x)}}\end{aligned}$$

IS estimator $Z(x) = e^{-\gamma S_{\tau(x)}}$

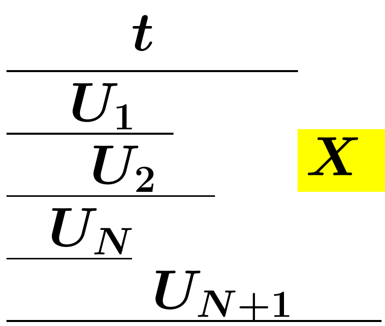
Bounded relative error

$\limsup \mathbb{V}\mathrm{ar} Z(x)/\mathbb{P}(S_N > x)^2 < \infty$

SA, Glynn p. 172; typos!!

Blanchet et al.

RESTART with $T \equiv t$, G exponential(μ)



$$X \;=\; t + U_1 + \cdots + U_N$$

$$\begin{aligned} 1 &= \int_0^t \mathrm{e}^{-\gamma(t)y} \mu \mathrm{e}^{-\mu y} \, \mathrm{d}y \\ &= \frac{\mu}{\mu - \gamma(t)} \big(1 - \mathrm{e}^{-(\mu - \gamma(t))t} \big) \end{aligned}$$

Algorithm 0

Simulate U's

as exponential$\big(\mu - \gamma(t)\big)$

truncated to $[0, t]$

Return $Z_0(x) = \mathrm{e}^{-\gamma(t) S_{\tau(x-t)}}$

Random T

$$\begin{aligned}
\overline{H}(x) &= \mathbb{P}(X > x) \\
&= \int_0^\infty \mathbb{P}\big(X > x \,\big|\, T = t\big) f(t)\,\mathrm{d}t \\
&\sim \int_0^\infty C(t)\mathrm{e}^{-\gamma(t)x} f(t)\,\mathrm{d}t \\
&\sim \int_{t_0}^\infty C(t)\mathrm{e}^{-\gamma(t)x} f(t)\,\mathrm{d}t \\
&\sim \int_{t_0}^\infty 1 \cdot \mathrm{e}^{-\mu\overline{G}(t)x} f(t)\,\mathrm{d}t
\end{aligned}$$

Close-to-diagonal:

assume F, G are connected
s.t. $y = \overline{G}(t)$ leads to
integral of Abelian/Tauberian type.

$$f(t) \;=\; g(t)\overline{G}(t)^{\beta-1} L_0\big(\overline{G}(t)\big)$$

F Gamma-like: $\overline{F}(x) \sim cx^{\alpha-1}\mathrm{e}^{-\lambda x}$
$g(t) = \mu\mathrm{e}^{-\mu t}$

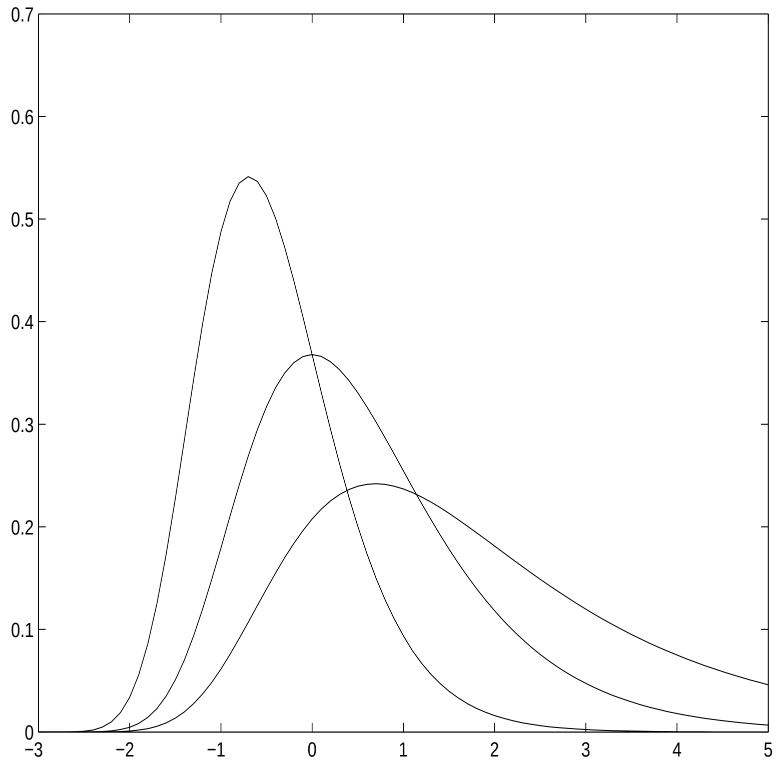
$\beta = \lambda/\mu$

$$\overline{H}(x) \sim \frac{c\Gamma(\beta)}{\mu^{\alpha+\beta}} \frac{\log^{\alpha-1} x}{x^{\beta}}$$

General principle for IS:
conditional distributions
RESTART:
T given $X > x$?

$$Y = \mu T - \log x - \log \mu \to Q_{\beta}$$

$q_{\beta}(y) = \exp\{-\mathrm{e}^{-y} - \beta y\}/\Gamma(\beta)$
$\beta = 1$: Gumbel (Fisher-Tippett)
$\beta \neq 1$: exponential tilting


0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
−3
−2
−1
0
1
2
3
4
5

Algorithm 1:

1) **Simulate** $\boldsymbol{Y = -\log\Gamma(\beta)}$
from $\boldsymbol{Q_\beta}$

2) **Let** $\boldsymbol{T = (Y + \log x + \log\mu)/\mu}$

3) **Given** $\boldsymbol{T = t}$, **use CMC**
for failures in $\boldsymbol{[0, x - t]}$

4) **Compute LR corrected**
estimator

Worse than CMC

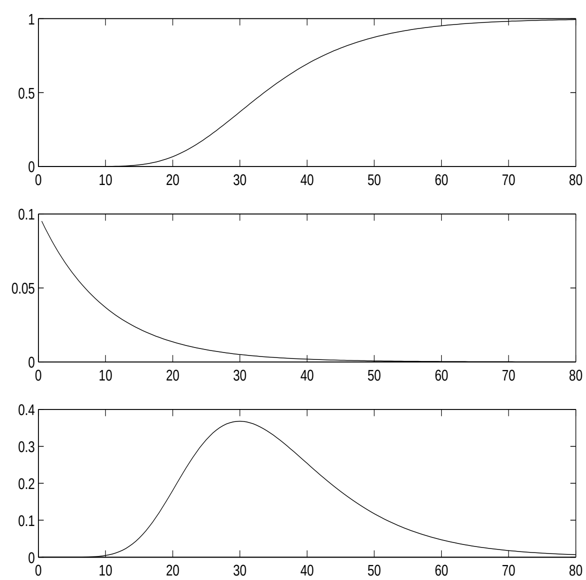


Figure 2:

Algorithm $1'$:

1) Simulate $Y = -\log\Gamma(\beta)$ from Q_β

2) Let $T = (Y + \log x + \log \mu)/\mu$

3') Given $T = t$, use Algorithm 0 $(\gamma(t))$ for failures in $[0, x - t]$

4) Compute LR corrected estimator

Bounded relative error

The Role of Rootfinding

$$1 = \int_0^t e^{\gamma(t)y} g(y)\,dy$$

Algorithm 0

1) Simulate U's from

$$e^{\gamma(t)y} g(y),\ 0 < y < t$$

2) Return $Z(x) = e^{-\gamma(t)S_{\tau(x-t)}}$

Set-up when $T \equiv t$

Each time if T random

Algorithm $0^{\#}$

1) Simulate U's from

$$g(y)/G(t),\ 0 < y < t$$

2) Return $Z_0^{\#} = G(t)^{\tau(x-t)}$

$$\mathbb{V}\mathrm{ar} Z_0^{\#} \sim \exp\{-(\gamma(t) + \xi(t))x\}$$

$$1 = G(t) \int_0^t e^{(\gamma(t)+\xi(t))y} g(y)\,dy$$

$\xi(t) < \gamma(t) \;\Rightarrow$ **not bd. rel. err.**
But $\xi(t) \sim \gamma(t)$ **as** $t \to \infty$

F **Gamma-like:** $\overline{F}(x) \sim c x^{\alpha-1} \mathrm{e}^{-\lambda x}$
$g(t) = \mu \mathrm{e}^{-\mu t}$
Algorithm $3^{\#}$**:**

1) **Simulate** $Y = -\log \Gamma(\beta)$
   from Q_β

2) **Let** $T = (Y + \log x + \log \mu)/\mu$

$3^{\#}$) **Given** $T = t$, **use**
   Algorithm $0^{\#}$
   for failures in $[0, x - t]$

4) **Compute LR corrected**
   estimator

Bounded relative error

Sheahan's exp-exp example
Algorithm $3^{\#}$, 10^3 replications
Runs in 50 seconds

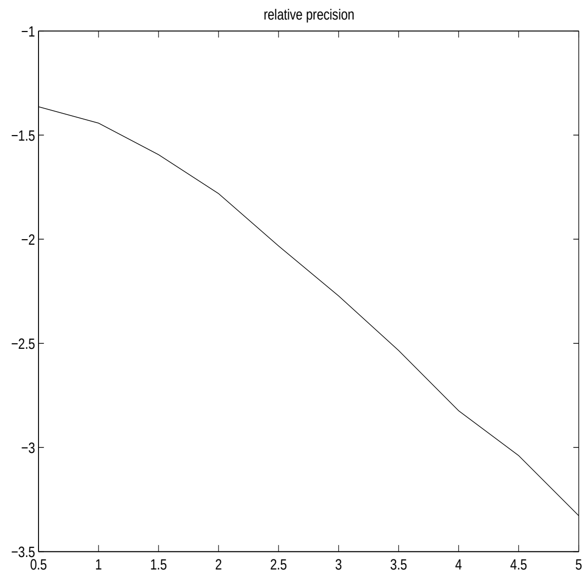


Figure 3:

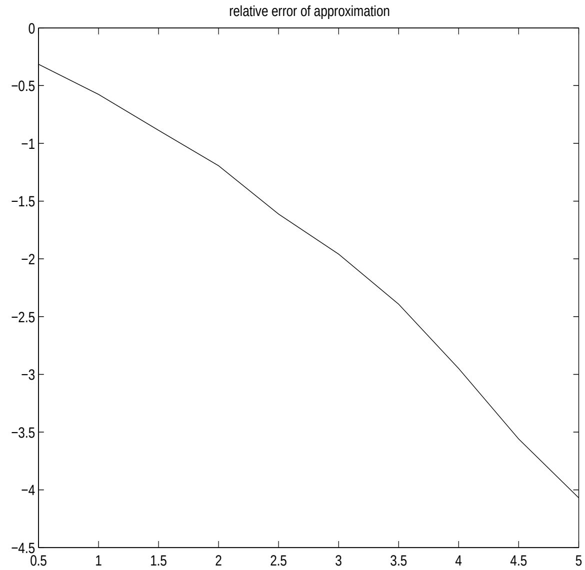

relative error of approximation
0
−0.5
−1
−1.5
−2
−2.5
−3
−3.5
−4
−4.5
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5

Figure 4:

T reg. varying

$f(t) = L(t)/t^{\alpha+1}$

$$\overline{H}(x) \sim \frac{L(\log x)\mu^{\alpha}}{\alpha \log^{\alpha} x}$$

Bare-hand calculations ...

$\mu T/\log x - 1$ has Pareto limit

Algorithm $2'$:

1) Simulate Y
from $\alpha/(1+y)^{\alpha+1}$

2) Let $T = \log x/\mu + \log xY/\mu$

3') Given $T = t$, use
CMC, Alg. 0 or $0^{\#}$
for failures in $[0, x-t]$

4) Compute LR corrected
estimator

Absolute continuity fails

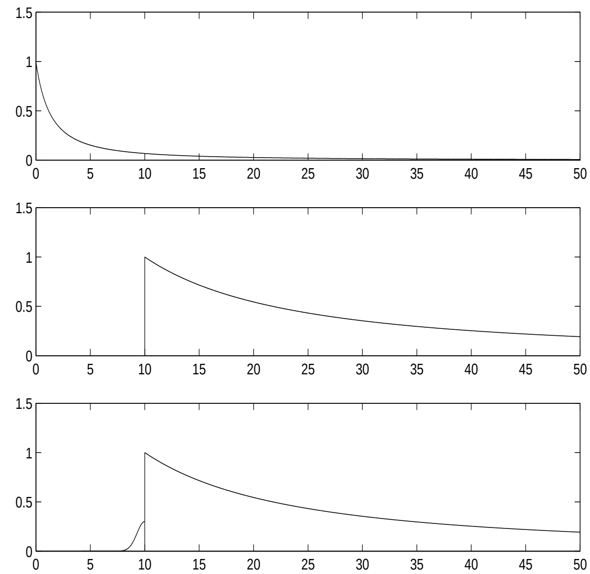


Figure 5:

Alg. 2′ only gives

$$\mathbb{P}(X > x, T < \log x/\mu)$$

$\mathbb{P}(X > x, T < \log x/\mu)$**?**

Separate simulation

$\log x - \mu T \mid X > x, T < \log x/\mu$

has limit with density

$$\mathrm{e}^{-\mathrm{e}^{z}}/E_1(1)$$

Algorithm $2''$:

1) **Simulate Y from $\mathrm{e}^{-\mathrm{e}^z}/E_1(1)$**

2) **Let $T = \log x/\mu - Y/\mu$**

3') **Given $T = t$, use Alg. 0 or $0^{\#}$ for failures in $[0, x - t]$**

4) **Compute LR corrected estimator**

Overall estimator: $Z_2'(x) + Z_2''(x)$

Bounded relative error

Running Time Issues

Variance σ^2/replication or rel. sq. error $\sigma^2/\mathbb{P}(X > x)^2$ too limited a criterion

Must consider also $\mathbb{E}$(CPU time) τ

Rel. sq. error / unit CPU time:

$$\frac{\tau\sigma^2}{\mathbb{P}(X > x)^2}$$

Present algorithms: $\tau \approx x$ (X is generated as sum of failures)

Present algorithms: $\tau \approx x$
(X is generated
as sum of failures)

Comparison with CMC

Generate X, return $1(X > x)$
sometimes $\tau = \infty$
(Robert Sheahan's problem?)
always improved efficiency

But: Could stop when sum $\geq x$
Then $\tau \leq x$
Picture less clear