Impulse problem on finite horizon with execution delay

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Motivations

Basic motivation: to study the pricing and hedging of an option on hedge fund.

- To buy or sell shares of hedge funds, the financial agents must declare their orders one or two months before they can be executed (because of the illiquid portfolio held by the hedge fund manager).

- The effective price is the price of the fund at the execution date.

- This is one of the aspects of illiquidity.

We study the impact of this lag in a general framework. Our goal will be to estimate the cost of liquidity.
Controlled diffusion model

- In the absence of control, let $X$ be a $\mathbb{R}^d$ valued process:

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t$$

- At any stopping time $\tau_i$, the agent may decide to intervene on the system with an impulsion $\xi_i \in E$ based on the information available at $\tau_i$.

- There is a minimum lag $h > 0$ between two interventions:

$$\tau_{i+1} - \tau_i \geq h.$$  

- This impulse takes effect, with delay, at time $\tau_i + mh$ with $m > 0 \in \mathbb{N}$. This moves the system to:

$$X_{\tau_i + mh} = \Gamma(X_{(\tau_i + mh)^-}, \xi_i)$$
Example

We can consider the following problem of optimal investment and/or indifference pricing:

- Let $S$ be the spot price of an hedge fund, with a Black-Scholes dynamic: \( \frac{dS_t}{S_t} = \mu dt + \sigma dW_t \)

- Let $C$ be the amount of cash in the agent’s portfolio, and $N$ be the number of shares of the hedge fund held by the agent.

- The impulsions $\xi_i$ represent the number of shares bought or sold by the agent at time $\tau_i$. 

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The system is described by the pending orders and the vector:

\[ X_t = (S_t, C_t, N_t) \]

When the order \( i \) is executed the system moves to:

\[
\begin{align*}
S_{\tau_i + mh} &= S_{(\tau_i + mh)^-} \\
C_{\tau_i + mh} &= C_{(\tau_i + mh)^-} - \xi_i S_{\tau_i + mh} \\
N_{\tau_i + mh} &= N_{(\tau_i + mh)^-} + \xi_i
\end{align*}
\]

The optimization problem can be, for example the maximization of the expected utility of the agent:

\[
\mathbb{E} \left[ U(C_T + N_T S_T - g(S_T)) \right]
\]
The goal is to solve the following problem:

$$v_0(0, x_0) = \sup_{\alpha \in A} \left\{ \mathbb{E} \left[ \int_0^T f(X^\alpha_t) \, ds + g(X^\alpha_T) \right] \right\}$$

with:

$$A = \{(\tau_i, e_i)_{i \in \mathbb{N}} \text{ s.t. } \tau_i \geq 0 \text{ is an } \mathbb{F} \text{ stopping time, } \tau_{i+1} \geq \tau_i + h \text{ and } e_i \text{ is } \mathcal{F}_{\tau_i} \text{-measurable} \forall i]\}.$$ 

This an impulse control problem with delay in finite horizon.
Related literature


An important issue is that the process $X_t$ is not Markovian by itself. Indeed, we must take the set of pending orders into account:

$$p = (t_i, \xi_i)_{i \in \{1..k\}} \in ([0, T] \times E)^k \text{ s.t. } t - mh < t_i \leq t \forall i.$$ 

In this framework, we have $k \leq m$, and therefore a finite dimensionnal system. This is why we introduced the minimum time $h$ between two interventions.
The set of possible pending orders at time $t$ is:

$$P_t(k) = \left\{ p = (t_i, \xi_i)_{i \in \{1..k\}} \in ([0, T] \times E)^k \text{ s.t.} \right. \\
\quad \left. t - mh < t_i \text{ and } t_i + h \leq t_{i+1} \leq t \forall i \right\}$$

with $k \leq m$.

We denote the consistent states as:

$$D_k = \left\{ (t, x, p) : (t, x) \in [0, T] \times \mathbb{R}^d, \ p \in P_t(k) \right\},$$

And the admissibles controls from a given set of pending orders:

$$A_{t,p} = \left\{ \alpha = (\tau_i, \xi_i)_{i \geq 1} \in \mathcal{A} : (\tau_i, \xi_i) = (t_i, e_i), \ i = 1, \ldots, k \\
\quad \text{and } \tau_{k+1} \geq t \right\},$$
The problem of the agent is to maximize the criterion:

\[ J_k(t, x, p, \alpha) = \mathbb{E} \left[ \int_t^T f(X^{t,x,p,\alpha}_s) ds + g(X^{t,x,p,\alpha}_T) \right], \]

for \((t, x, p) \in D_k, k = 0, \ldots, m, \alpha = (\tau_i, \xi_i)_{i \in A_{t,p}}\).

We get the corresponding value functions:

\[ v_k(t, x, p) = \sup_{\alpha \in A_{t,p}} J_k(t, x, p, \alpha), \]

for \(k = 0, \ldots, m, (t, x, p) \in D_k\),
Dynamic programming principle: no possible action

- We introduce the set:
\[
\mathcal{D}^1_k = \{(t, x, p) : (t, x) \in [0, T] \times \mathbb{R}^d, \ p \in P_t(k) \text{ s.t. } t_k > t - h\}
\]
- If \((t, x, p) \in \mathcal{D}^1_k\) there is no immediate action possible:
\[
v_k(t, x, p) = \mathbb{E} \left[ \int_t^\theta f(X^{t, x, 0}_s) ds + v_k(\theta, X^{t, x, 0}_\theta, p) \right].
\]
- Only the diffusion of \(X\) operates, and the associated PDE is:
\[
- \frac{\partial v_k}{\partial t} - \mathcal{L}v_k - f(\cdot) = 0, \quad (1)
\]
on \(\mathcal{D}^{1,m}_k\).
- Note that for \(v_m\), no more orders can be passed until the first execution. All the consistent states of \(v_m\) lie in \(\mathcal{D}^{1,m}_m\).
Dynamic programming principle: possible intervention

- We introduce the set:
  \[
  \mathcal{D}_k^2 = \{(t, x, p) : (t, x) \in [0, T] \times \mathbb{R}^d, p \in P_t(k) \text{ s.t. } t_k \leq t - h\}
  \]

- If \((t, x, p) \in \mathcal{D}_k^2\) the agent can take action in \([t, t + dt]\). He has the choice between letting the diffusion of \(X\) operate, and passing an order.

- If he passes an order, the value function jumps from \(v_k(t, x, p)\) to \(v_{k+1}(t, x, p \cup (t, \xi))\). We get that
  \[
  v_k(t, x, p) \geq \sup_{\xi} \{v_{k+1}(t, x, p \cup (t, \xi))\}
  \]

- This corresponds to the variationnal inequality on \(\mathcal{D}_k^2\):
  \[
  \min \left\{ -\frac{\partial v}{\partial t} - \mathcal{L}v_k - f(.) , v_k(t, x, p) - \sup_{e \in E} \{v_{k+1}(t, x, p \cup (t, e))\} \right\} = 0
  \]
Boundary conditions

- The terminal condition is, at time $T$:

$$\lim_{(\tilde{t},\tilde{x},\tilde{p}) \to (T,x,p)} v_k(\tilde{t},\tilde{x},\tilde{p}) = g(x) \text{ for all } (x,p) \in D_k^m$$  \hspace{1cm} (3)

- But we must also prove the boundary condition linked to the execution of the first pending order, at time $t_1 + mh < T$:

$$\lim_{(\tilde{t},\tilde{x},\tilde{p}) \to (t_1+mh,x,p)} v_k(\tilde{t},\tilde{x},\tilde{p}) = v_{k-1}(t_1 + mh, \Gamma(x,e), p \setminus (t_1, e_1))$$

for all $(x,p) \in D_k^m$, $k \geq 1$ s.t. $p = (t_1, e_1) \cup (t_i, e_i)_{i \in 2..k}$. This is more difficult due to continuity issues for $v_{k-1}$. 

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The main theorem

**Theorem**

The family of value functions $v_k$, $k = 0, \ldots, m$, is the unique viscosity solution to (1) on $D^1_k$ and (2) on $D^2_k$, which satisfy the boundary data (3)-(4), a linear growth condition, and the condition:

$$v_k(t_k + h, x, p) \geq \sup_{e \in E} \{v_{k+1}(t_k + h, x, p \cup (t_k + h, e))\}.$$ 

Moreover, $v_k$ is continuous on $D_k$. 
A nonstandard problem

To compute the value functions $v_k$ for all $k$, we face an issue:

- $v_k$ depends on $v_{k+1}$ via the characteristic PDE on $\mathcal{D}_{k}^{2,m}$, because of the possibility of passing an order.

- $v_{k+1}$ depends on $v_k$ via the boundary condition. It corresponds to the execution of the first pending order of $v_{k+1}$.

This problem is solved with the following algorithm.
Initialization

The first step of the algorithm is based on the following remark:

- If an order is passed after $T - mh$, it will be executed after $T$. Therefore it has no action on $X_t$ for $t \leq T$.
- Hence, if $p$ is such that $t_1 > T - mh$ (i.e. all the pending orders were passed after $T - mh$), the value function is such that:

$$v_k(t, x, p) = \mathbb{E}\left[ \int_t^T f(X_s^{t,x,0})ds + g(X_T^{t,x,0}) \right],$$

which is easily computable.
Known calculations at step $n$

- We define the following set, for $k \in \{1..m\}$:

$$
\mathcal{D}_k(n) = \{(t, x, p) \in \mathcal{D}_k \text{ s.t. } t_1 \geq T - nh\},
$$

which represents the consistent states such that the first pending order was passed after $T - nh$.

- The hypothesis of step $n$ is that we know $\nu_k$ on $\mathcal{D}_k(n)$ for all $k \in \{1..m\}$, and $\nu_0$ for all $t \geq T - nh$. 

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From step $n$ to $n+1$

One can calculate $v_m$ on $\mathcal{D}_k(n+1) \setminus \mathcal{D}_k(n)$ for $t \in [t_m, t_1 + mh]$. If $t_1 \in [T - (n + 1)h, T - nh)$, then $t_2 \geq T - nh$.

- Indeed, the PDE satisfied by $v_m$ on $\mathcal{D}_k(n+1) \setminus \mathcal{D}_k(n)$ is:

$$- \frac{\partial v_m}{\partial t} - \mathcal{L} v_m - f(.) = 0,$$

- The boundary condition of $v_m$ is:

$$\lim_{(\tilde{t}, \tilde{x}, \tilde{p}) \rightarrow (t_1 + mh, x, p)} v_m(\tilde{t}, \tilde{x}, \tilde{p}) = v_{m-1}(t_1 + mh, \Gamma(x, e), p \setminus (t_1, e_1))$$

with $(t_1 + mh, \Gamma(x, e), p \setminus (t_1, e_1)) \in \mathcal{D}_k(n)$. This has been calculated at step $n$. 
From step $n$ to $n + 1$

For fixed $n$, we proceed with a backward induction on $k$.

- If $v_{k+1}$ has been calculated on $D_{k+1}(n + 1) \setminus D_{k+1}(n)$. Then $v_k$ can be calculated on $D_k(n + 1) \setminus D_k(n)$. Indeed, the terminal condition:

$$
\lim_{(\tilde{t}, \tilde{x}, \tilde{p}) \to (t_1 + mh, x, p)} v_{k+1}(\tilde{t}, \tilde{x}, \tilde{p}) = v_k(t_1, \Gamma(x, e_1), p \setminus (t_1, e_1))
$$

has been calculated at step $n$. And the characteristic PDE involves only the knowledge of $v_{k+1}$ on $D_{k+1}(n + 1) \setminus D_{k+1}(n)$. 
Conclusion

- We have a method to solve a large class of control problems with delay.
- The drawback is that the dimension of the problem grows quickly with $m$. Typically have to solve a PDE of dimension $\dim X + 2m$.
- In most cases it is numerically tracktable for $m = 1$ or $m = 2$. 