Impulse problem on finite horizon with execution delay

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Motivations

Basic motivation: to study the pricing and hedging of an option on hedge fund.

- To buy or sell shares of hedge funds, the financial agents must declare their orders one or two months before they can be executed (because of the illiquid portfolio held by the hedge fund manager).
- The effective price is the price of the fund at the execution date.
- This is one of the aspects of illiquidity.

We study the impact of this lag in a general framework. Our goal will be to estimate the cost of liquidity.

Controlled diffusion model

- In the absence of control, let X be a $\mathbb{R}^d$ valued process:

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t$$

- At any stopping time τ_i, the agent may decide to intervene on the system with an impulsion $\xi_i \in E$ based on the information availaible at τ_i.
- There is a minimum lag $h > 0$ between two interventions: $\tau_{i+1} - \tau_i \geq h$.
- This impulse takes effect, with delay, at time $\tau_i + mh$ with $m > 0 \in \mathbb{N}$. This moves the system to:

$$X_{\tau_i+mh} = \Gamma(X_{(\tau_i+mh)^-}, \xi_i)$$

Example

We can consider the following problem of optimal investment and/or indifference pricing:

- Let S be the spot price of an hedge fund, with a Black-Scholes dynamic: $\frac{dS_t}{S_t} = \mu dt + \sigma dW_t$
- Let C be the amount of cash in the agent's portfolio, and N be the number of shares of the hedge fund held by the agent.
- The impulsions ξ_i represent the number of shares bought or sold by the agent at time τ_i.

Example

- The system is described by the pending orders and the vector:

$$X_t = (S_t, C_t, N_t)$$

- When the order i is executed the system moves to:

$$\begin{aligned} S_{\tau_i+mh} &= S_{(\tau_i+mh)^-} \\ C_{\tau_i+mh} &= C_{(\tau_i+mh)^-} - \xi_i S_{\tau_i+mh} \\ N_{\tau_i+mh} &= N_{(\tau_i+mh)^-} + \xi_i \end{aligned}$$

- The optimization problem can be, for example the maximization of the expected utility of the agent:

$$\mathbb{E}\left[U(C_T + N_T S_T - g(S_T))\right]$$

Problem formulation

The goal is to solve the following problem:

$$v_0(0, x_0) = \sup_{\alpha \in \mathcal{A}} \left\{ \mathbb{E}\Big[\int_0^T f(X_t^\alpha)ds + g(X_T^\alpha) \Big] \right\}$$

with:

$$\mathcal{A} = \{(\tau_i, e_i)_{i \in \mathbb{N}} \text{ s.t. } \tau_i \geq 0 \text{ is an } \mathbb{F} \text{ stopping time, } \tau_{i+1} \geq \tau_i + h \\ \text{and } e_i \text{ is } \mathcal{F}_{\tau_i}\text{-measurable} \forall i\} .$$

This an impulse control problem with delay in finite horizon.

Related literature

- Bensoussan A. and J.L. Lions (1982) : *Contrôle impulsionnel et inéquations variationnelles*, Dunod.
- Bar-Ilan A. and A. Sulem (1995) : "Explicit solution of inventory problems with delivery lags", *Math. Oper. Res.*, **20**, 709-720.
- Oksendal B. and A. Sulem (2006) : "Optimal stochastic impulse control with delayed reaction", Preprint, University of Oslo.

Markovian setting

- An important issue is that the process X_t is not Markovian by itself. Indeed, we must take the set of pending orders into account:

$$p = (t_i, \xi_i)_{i \in \{1..k\}} \in ([0, T] \times E)^k \text{ s.t. } t - mh < t_i \leq t \forall i.$$

- In this framework, we have $k \leq m$, and therefore a finite dimensionnal system. This is why we introduced the minimum time h between two interventions.

Some notations

- The set of possible pending orders at time t is:

$$P_t(k) = \Big\{ p = (t_i, \xi_i)_{i \in \{1..k\}} \in ([0, T] \times E)^k \text{ s.t.} \\ t - mh < t_i \text{ and } t_i + h \leq t_{i+1} \leq t \forall i \Big\}$$

  with $k \leq m$.
- We denote the consistent states as:

$$\mathcal{D}_k = \Big\{ (t, x, p) \; : (t, x) \in [0, T] \times \mathbb{R}^d, \; p \in P_t(k) \Big\},$$

- And the admissibles controls from a given set of pending orders:

$$\mathcal{A}_{t,p} = \Big\{ \alpha = (\tau_i, \xi_i)_{i \geq 1} \in \mathcal{A} \; : (\tau_i, \xi_i) = (t_i, e_i), \; i = 1, \ldots, k \\ \text{and } \; \tau_{k+1} \geq t \Big\},$$

Value function

- The problem of the agent is to maximize the criterion:

$$J_k(t,x,p,\alpha) = \mathbb{E}\Big[\int_t^T f(X_s^{t,x,p,\alpha})ds + g(X_T^{t,x,p,\alpha})\Big],$$

  for $(t,x,p) \in \mathcal{D}_k$, $k = 0,\dots,m$, $\alpha = (\tau_i,\xi_i)_i \in \mathcal{A}_{t,p}$.
- We get the corresponding value functions:

$$v_k(t,x,p) = \sup_{\alpha \in \mathcal{A}_{t,p}} J_k(t,x,p,\alpha),$$

  for $k = 0,\dots,m$, $(t,x,p) \in \mathcal{D}_k$,

Dynamic programming principle: no possible action

- We introduce the set:

$$\mathcal{D}_k^1 = \left\{(t,x,p) \ : (t,x) \in [0,T] \times \mathbb{R}^d, \ p \in P_t(k) \text{ s.t. } \ t_k > t - h\right\}$$

- If $(t,x,p) \in \mathcal{D}_k^1$ there is no immediate action possible:

$$v_k(t,x,p) = \mathbb{E}\Big[\int_t^\theta f(X_s^{t,x,0})ds + v_k(\theta, X_\theta^{t,x,0}, p)\Big].$$

- Only the diffusion of X operates, and the associated PDE is:

$$-\frac{\partial v_k}{\partial t} - \mathcal{L}v_k - f(.) = 0, \qquad (1)$$

on $\mathcal{D}_k^{1,m}$.

- Note that for v_m, no more orders can be passed until the first execution. All the consistent states of v_m lie in $\mathcal{D}_m^{1,m}$.

Dynamic programming principle: possible intervention

- We introduce the set:

$$\mathcal{D}_k^2 = \left\{(t,x,p)\ : (t,x) \in [0,T] \times \mathbb{R}^d,\ p \in P_t(k) \text{ s.t. } t_k \leq t-h\right\}$$

- If $(t,x,p) \in \mathcal{D}_k^{2,m}$ the agent can take action in $[t, t+dt]$. He has the choice between letting the diffusion of X operate, and passing an order.
- If he passes an order, the value function jumps from $v_k(t,x,p)$ to $v_{k+1}(t,x,p \cup (t,\xi))$. We get that $v_k(t,x,p) \geq \sup_\xi \left\{v_{k+1}(t,x,p \cup (t,\xi))\right\}$
- This corresponds to the variationnal inequality on $\mathcal{D}_k^{2,m}$:

$$\min\left\{-\frac{\partial v}{\partial t} - \mathcal{L}v_k - f(.), v_k(t,x,p) - \sup_{e\in E}\left\{v_{k+1}(t,x,p \cup (t,e))\right\}\right\}$$

$$= 0$$

Boundary conditions

- The terminal condition is, at time T:

$$\lim_{(\tilde{t},\tilde{x},\tilde{p})\to(T,x,p)} v_k(\tilde{t},\tilde{x},\tilde{p}) = g(x) \text{ for all } (x,p)\in\mathcal{D}_k^m \qquad (3)$$

- But we must also prove the boundary condition linked to the execution of the first pending order, at time $t_1 + mh < T$:

$$\lim_{(\tilde{t},\tilde{x},\tilde{p})\to(t_1+mh,x,p)} v_k(\tilde{t},\tilde{x},\tilde{p}) = v_{k-1}(t_1+mh,\Gamma(x,e),p\setminus(t_1,e_1))$$

for all $(x,p)\in\mathcal{D}_k^m$, $k\geq 1$ s.t. $p=(t_1,e_1)\cup(t_i,e_i)_{i\in 2..k}$. This is more difficult due to continuity issues for v_{k-1}.

The main theorem

Theorem

The family of value functions v_k, $k = 0, \ldots, m$, is the unique viscosity solution to (1) on $\mathcal{D}_k^1$ and (2) on $\mathcal{D}_k^2$, which satisfy the boundary data (3)-(4), a linear growth condition, and the condition:

$$v_k(t_k + h, x, p) \geq \sup_{e \in E} \left\{ v_{k+1}(t_k + h, x, p \cup (t_k + h, e)) \right\}.$$

Moreover, v_k is continuous on $\mathcal{D}_k$.

A nonstandard problem

To compute the value functions v_k for all k, we face an issue:

- v_k depends of v_{k+1} via the characteristic PDE on $\mathcal{D}_k^{2,m}$, because of the possibility of passing an order.
- v_{k+1} depends of v_k via the boundary condition. It corresponds to the execution of the first pending order of v_{k+1}.

This problem is solved with the following algorithm.

Initialization

The first step of the algorithm is based one the following remark:

- If an order is passed after $T - mh$, it will be executed after T. Therefore it has no action on X_t for $t \leq T$.
- Hence, if p is such that $t_1 > T - mh$ (i.e. all the pending orders were passed after $T - mh$), the value function is such that:

$$v_k(t, x, p) = \mathbb{E}\Big[\int_t^T f(X_s^{t,x,0}) ds + g(X_T^{t,x,0}) \Big],$$

which is easely computable.

Known calculations at step n

- We define the following set, for $k \in \{1..m\}$:

  $$\mathcal{D}_k(n) = \{(t, x, p) \in \mathcal{D}_k \text{ s.t. } t_1 \geq T - nh\},$$

  which reprensents the consitent states such that the first pending order was passed after $T - nh$.
- The hypothesis of step n is that we know v_k on $\mathcal{D}_k(n)$ for all $k \in \{1..m\}$, and v_0 for all $t \geq T - nh$.

From step n to $n+1$

One can calculate v_m on $\mathcal{D}_k(n+1) \setminus \mathcal{D}_k(n)$ for $t \in [t_m, t_1 + mh]$. If $t_1 \in [T-(n+1)h, T-nh)$, then $t_2 \geq T - nh$.

- Indeed, the PDE satisfied by v_m on $\mathcal{D}_k(n+1) \setminus \mathcal{D}_k(n)$ is:

$$-\frac{\partial v_m}{\partial t} - \mathcal{L} v_m - f(.) = 0,$$

- The boundary condition of v_m is:

$$\lim_{(\tilde{t},\tilde{x},\tilde{p}) \to (t_1+mh,x,p)} v_m(\tilde{t},\tilde{x},\tilde{p}) = v_{m-1}(t_1+mh, \Gamma(x,e), p \setminus (t_1,e_1))$$

with $(t_1 + mh, \Gamma(x,e), p \setminus (t_1, e_1)) \in \mathcal{D}_k(n)$. This has been calculated at step n.

From step n to $n+1$

For fixed n, we proceed with a backward induction on k.

- If v_{k+1} has been calculated on $\mathcal{D}_{k+1}(n+1) \setminus \mathcal{D}_{k+1}(n)$. Then v_k can be calculated on $\mathcal{D}_k(n+1) \setminus \mathcal{D}_k(n)$. Indeed, the terminal condition :

$$\lim_{(\tilde{t},\tilde{x},\tilde{p}) \to (t_1+mh,x,p)} v_{k+1}(\tilde{t},\tilde{x},\tilde{p}) = v_k(t_1,\Gamma(x,e_1),p \setminus (t_1,e_1))$$

has been calculated at step n. And the characteristic PDE involves only the knowledge of v_{k+1} on $\mathcal{D}_{k+1}(n+1) \setminus \mathcal{D}_{k+1}(n)$.

Conclusion

- We have a method to solve a large class of control problems with delay.
- The drawback is that the dimension of the problem grows quickly with m. Typically have to solve a PDE of dimension $\dim X + 2m$.
- In most cases it is numerically tracktable for $m = 1$ or $m = 2$.