Risk neutral dynamics of spot and forward electricity prices

- Joint work with J.-M. Marin & N. Touzi -

Luciano Campi

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Introduction

The Model

Electricity forward prices

Constant coefficients

What’s next

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Introduction I : Motivations

- In standard financial markets: $F_t(T) = S_t e^{r(T-t)}$. This equality relies heavily on costless storability of financial assets, it breaks down when $S_t$ is spot price of electricity.

- A priori, no relations between spot and forward at least in a market composed of electricity and bank account (see, e.g., Geman-Vasicek (2001)).

- Geman-Vasicek (2001) and Bessembinder-Lemon (2002) show that short-term forward contract are (upward- or downward-) biased estimator of spot prices, so ...

- ... in mathematical terms, when $t \uparrow T$, $F_t(T)$ does not necessarily tend to $S_T$. 

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Nevertheless, imagine an fictitious economy where electricity is produced only out of coal, so that electricity spot price $P_t = c_c S_t^c$, and agents can trade coal, buy electricity and have a bank account.

Assume no-arbitrage in the market of coal and bank account, i.e. there exists a risk-neutral measure $Q$ for \( \tilde{S}_t^c = e^{-rt} S_t^c \).

A forward contract on spot electricity $P_T$ can be viewed as a contract on coal necessary to produce 1 MWh of electricity, with price $c_c S_t^c$, so that

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The Model

- **Randomness**: \((W^0, W) = (W^0, W^1, \ldots, W^n)\) Wiener process defined on a given \((\Omega, \mathcal{F}, \mathbb{P})\) and \(\mathcal{F}_t = \mathcal{F}_t^0 \vee \mathcal{F}_t^W\) models the market information flow.

- Riskless asset \(S^0_t = \exp \int_0^t r_u du, t \geq 0, r\) is \(\mathcal{F}_t^0\)-adapted and \(\geq 0\).

- **Commodities market**: \(n \geq 1\) commodities (coal, gas, ...) whose prices \(S^i\) to produce 1 MWh of electricity follows

\[
dS^i_t = S^i_t (\mu^i_t dt + \sum_j \sigma^i_j dW^j_t), \quad t \geq 0.
\]

For simplicity, assume that convenience yields \(y^i = 0\) for all \(i = 1, \ldots, n\).

- **Electricity demand**: \(D = (D_t)_{t \geq 0}\) \(\mathcal{F}_t^0\)-adapted, positive process; notice that \(D\) is independent of each \(S^i\).
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- **Electricity demand**: \(D = (D_t)_{t \geq 0}\) \(\mathcal{F}^0_t\)-adapted, positive process; notice that \(D\) is independent of each \(S^i\).
Electricity spot price $P_t$ when all technologies are available:

- $\Delta_i > 0$ denotes the capacity of $i$-th commodity for electricity at every instant, a constant known to the producer.

- Order commodities prices $S_t^{(1)}(\omega) \leq \ldots \leq S_t^{(n)}(\omega)$ from the cheapest to the most expensive, giving an $\mathcal{F}_t^W$-adapted random permutation $\pi_t(\omega)$ of $\{1, \ldots, n\}$.

- Look at the demand $D_t$:

$$D_t \in l^\pi_{\pi_t} := \left[ \sum_{i=1}^{k-1} \Delta_{\pi_t(i)}, \sum_{i=1}^{k} \Delta_{\pi_t(i)} \right] \Rightarrow P_t = S_t^{(k)}$$

- ... so that $P_t = \sum_k S_t^{(k)} 1_{l^\pi_{\pi_t}}(D_t)$ for $t \geq 0$. 
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Technologies failures I: The case of two commodities

If $n=2$ we have $S^1_t \leq S^2_t$ or $S^2_t \leq S^1_t$, let's consider the first case $\pi_t = \{1, 2\}$.

Introduce two r.v.'s $\epsilon^i_t$, $i = 1, 2$ such that

- $\epsilon^i_t = 1$ when technology $i$ is available, otherwise $\epsilon^i_t = 0$
- $\epsilon^i_t = 0$ implies that $\epsilon^j_t = 1$ for $i \neq j$

Only three cases may happen at each time $t$

1. $\epsilon^1_t = \epsilon^2_t = 1$ then $P_t = S^1_t 1_{[0,\Delta_1]}(D_t) + S^2_t 1_{[\Delta_1,\Delta_1+\Delta_2]}(D_t)$
2. $\epsilon^1_t = 1$, $\epsilon^2_t = 0$ then $P_t = S^1_t 1_{[0,\Delta_1]}(D_t)$
3. $\epsilon^1_t = 0$, $\epsilon^2_t = 1$ then $P_t = S^2_t 1_{[0,\Delta_1+\Delta_2]}(D_t)$

To sum up:

$$P_t = S^1_t 1_{[0,\Delta_1 \epsilon^1_t]}(D_t) + S^2_t 1_{[\Delta_1 \epsilon^1_t,\Delta_1 \epsilon^1_t+\Delta_2 \epsilon^2_t]}(D_t)$$
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Technologies failures II : The general case

- Let $\eta$ be a new process, values in $\{0, 1, \ldots, n\}$, with interpretation
  - event $\{\eta_t = i\}$ means “$i$-th technology not available”, for $1 \leq i \leq n$
  - event $\{\eta_t = 0\}$ means that all technologies are available
  
  *Hidden assumption: only one failure at the time is allowed.*

- Define $\epsilon^i_t := 1_{\{\eta_t \neq i\}}$, $1 \leq i \leq n$, so that
  - $\epsilon^i_t = 1$ means “$i$-th technology available at time $t$"
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- Set $l^{\pi_t}_k(t) := \left[\sum_{i=1}^{k-1} \Delta \pi_t(i) \epsilon^i_t \pi_t(i), \sum_{i=1}^k \Delta \pi_t(i) \epsilon^i_t \pi_t(i)\right]

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- ... so that \( P_t = \sum_k S_t^{(k)} 1_{l^\pi_k(t)} \) for \( t \geq 0 \).
No-arbitrage assumption on commodities market.

Let $T > 0$. There exists $\mathbb{Q} \sim \mathbb{P}$ on $\mathcal{F}_T^W$ such that:

1. Each $\tilde{S}_t^i/S_0$ is a $\mathbb{Q}$-martingale w.r.t. $\mathcal{F}_t^W$.
2. The laws of $W^0$ and $\eta$ do not change.
3. Filtrations $(\mathcal{F}_t^0), (\mathcal{F}_t^W), (\mathcal{F}_t^\eta)$ are $\mathbb{Q}$-independent.

Remarks

1. Property 3 above is satisfied if $W^0$, $W$ and $\eta$ are constructed on the canonical product space and the change of measure affects only the factor where $W$ is defined.
2. Being $D$ not tradable, this market is not complete. We choose $\mathbb{Q}$ as the pricing measure.
The pay-off of a forward contract on spot electricity is

\[ P_T = \sum_k S_T^{(k)} \mathbf{1}_{\pi T} (D_T) \] so it can be viewed as an option on commodities.

Use no-arbitrage assumption on commodities to get

\[
F_t(T) = \mathbb{E}^{Q_T}[P_T | \mathcal{F}_t] = \mathbb{E}^{Q_T} \left[ \sum_{k=1}^{n} S_T^{(k)} \mathbf{1}_{\pi T} (D_T) | \mathcal{F}_t \right]
\]

where \( Q_T \) is the forward risk-neutral measure on \( \mathcal{F}_T \):

\[
\frac{dQ_T}{dQ} = \frac{\exp \int_t^T r_u du}{\mathbb{E}^Q[\exp \int_t^T r_u du | \mathcal{F}_t]}
\]

(Notice that \( Q_T = Q \) if \( r \) is non-random)
The pay-off of a forward contract on spot electricity is

\[ P_T = \sum_k S_T^{(k)} 1_{l_k^T(T)}(D_T) \]

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Use no-arbitrage assumption on commodities to get

\[ F_t(T) = \mathbb{E}^{Q_T}[P_T|F_t] = \mathbb{E}^{Q_T}\left[ \sum_{k=1}^{n} S_T^{(k)} 1_{l_k^T(T)}(D_T)|F_t \right] \]

where \( Q_T \) is the forward risk-neutral measure on \( F_T \):

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Electricity forward prices II : The main formula

Proposition

Under previous assumptions and if $S^i_T \in L^1(\mathbb{Q}_T)$, $1 \leq i \leq n$ : for all $t \in [0, T]$

$$F_t(T) = \sum_{i=1}^{n} \sum_{\pi \in \Pi_n} c_{\pi(i)} F_{\pi(i)}^i(T) \mathbb{Q}_T[D_T \in I_{\pi(i)}^i(T)|\mathcal{F}_t^0]$$

$$\times \mathbb{Q}_T^{\pi(i)}[\pi_T = \pi|\mathcal{F}_t^W]$$

where :

- $\Pi_n$ is the set of all permutations of $\{1, \ldots, n\}$
- $F^i_t(T)$ is forward price of $i$-th commodity, delivery date $T$
- $d\mathbb{Q}_T^{\pi(i)}/d\mathbb{Q}_T = S_T^{\pi(i)}/\mathbb{E}_\mathbb{Q}_T[S_T^{\pi(i)}]$ on $\mathcal{F}_t^W$
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**Proposition**

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- \( d\mathbb{Q}_T^{\pi(i)}/d\mathbb{Q}_T = S_T^{\pi(i)}/\mathbb{E}\mathbb{Q}_T[S_T^{\pi(i)}] \) on \( \mathcal{F}_T^W \)
Electricity forward prices II: The main formula

Proposition

Under previous assumptions and if \( S^i_T \in L^1(\mathbb{Q}_T), \ 1 \leq i \leq n : \) for all \( t \in [0, T] \)

\[
F_t(T) = \sum_{i=1}^{n} \sum_{\pi \in \Pi_n} c_{\pi(i)} F^\pi(i)(T) \mathbb{Q}_T[D_T \in l^\pi_i(T)|\mathcal{F}_t^0] \\
\times \mathbb{Q}_T^{\pi(i)}[\pi_T = \pi|\mathcal{F}_t^W]
\]

where:

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- \( d\mathbb{Q}_T^{\pi(i)}/d\mathbb{Q}_T = S^\pi_T/\mathbb{E}^{\mathbb{Q}_T}[S^\pi_T] \) on \( \mathcal{F}_T^W \)
Electricity forward prices III: some remarks

This model is able to explain three basic features of electricity market as:

- **Observed spikes in electricity spot prices dynamics**
- **Non-convergence of electricity forward prices towards spot (day-ahead) prices as** $t \uparrow T$. Indeed,

$$F_t(T) \rightarrow F_T(T) = \sum_{i=1}^{n} S_T^{(i)} \mathbb{Q}_T[x \in l_i^\pi(T)]|_{x=D_T,\pi=\pi_T}.$$  

$$F_T(T) \neq P_T$$ whenever $\eta$ is non-degenerate.

- **The paths of electricity forward prices are much smoother than the corresponding spot prices.**
The constant coefficients model: more explicit formulae

- Commodities prices $S^i$ follow $n$-dim Black-Scholes model: volatilities $\sigma_{ij} > 0$ and interest rate $r > 0$ constant so that, in particular, $Q_T = Q$

- $F_t^i(T) = e^{r(T-t)}S_t^i$ for all commodities $1 \leq i \leq n$

- Demand of electricity: $D$ follows a OU process

\[
dD_t = a(b - D_t)dt + \delta dW^0_t, \quad D_0 > 0
\]

with $a, b, \delta > 0$.

- Under these assumptions probabilities $Q[D_T \in I_k^\pi(T) | \mathcal{F}^0_t]$ and $Q_T^{\pi(i)}[\pi_T = \pi | \mathcal{F}^W_t]$ can be computed explicitly as functions of the parameters.
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Under these assumptions probabilities $Q[D_T \in \lambda^i_k(T) | F^0_t]$ and $Q^\pi_T[\pi_T = \pi | F^W_t]$ can be computed explicitly as functions of the parameters.
What’s next?

- Pricing of options on forward electricity: it can be reduced to pricing of basket options of commodities
- Simulations and estimation of parameters in progress...
- Make the model more complex, e.g. add stochastic convenience yields and interest rate, more than one failure at the time...
- Study the risk premium $\pi(t, T) = F_t(T) - P_t$ in our model, compare with other models
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