Variable length memory chains and regeneration

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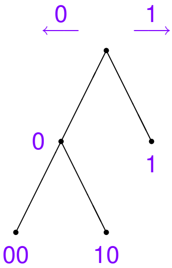
supervisor: Antonio Galves

MAS 2008, Rennes

An example of tree



Finite alphabet $A = \{0, 1\}$

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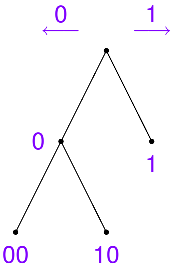

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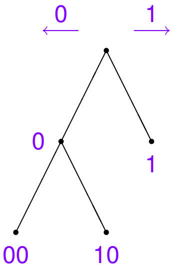
Let call this tree τ.

It can be identified with the set

of its leaves: $\tau = \{00, 10, 1\}$.

These leaves are called contexts.

Definition of a context tree



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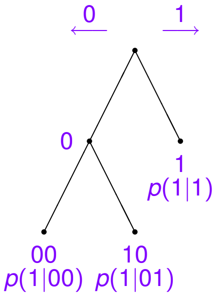
Definition of a probabilistic context tree



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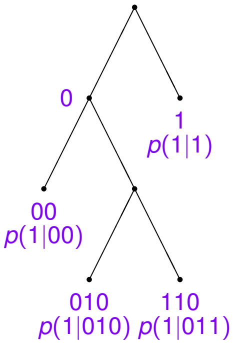
- A probabilistic context tree is a pair (τ, p) where p is a set of transition probabilities: $p = \{p(1|\omega), \omega \in \tau\}$.

Let's construct a stochastic chain with (τ, p)



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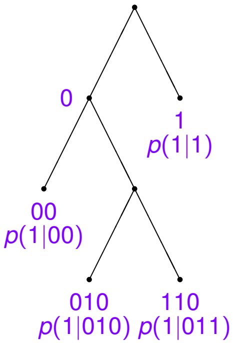

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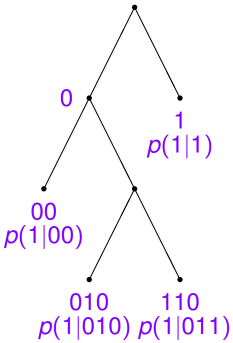

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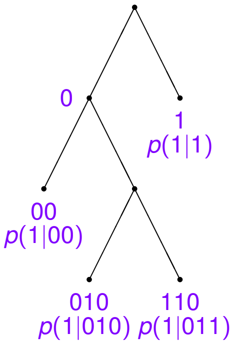

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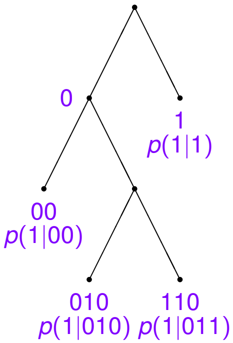

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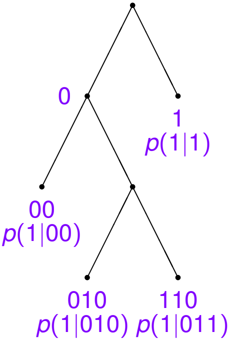

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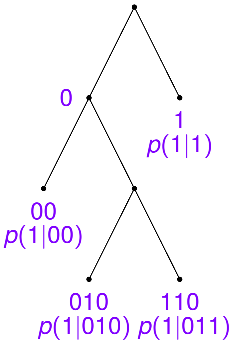

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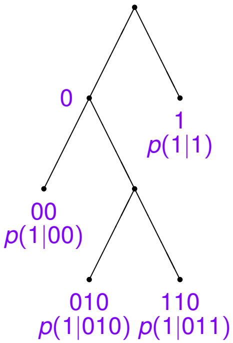

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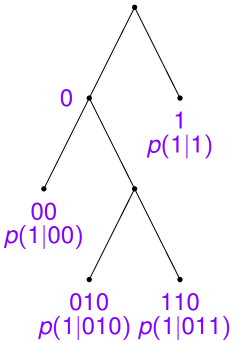

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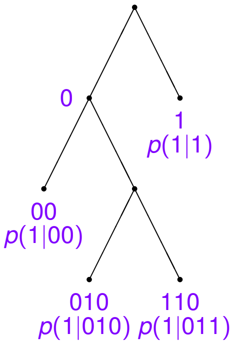

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..etc... that is the way we construct a chain

with variable lenght memory.

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REMARK that the finite size model is nothing else than a "parcimonieuse" Markov chain, thus we know everything about existence, uniqueness and perfect simulation of the stationary measure.

Some references

On regeneration and prefect simulation for Markov chains

- J. G. Propp, D. B. Wilson. Exact sampling with coupled Markov chains and applications to statistical mechanics. *Random Structures and Algorithms*, **9**, 223-252, 1996.

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On Variable length Markov chains

- J. Rissanen. A universal data compression system. *IEEE Trans. Inform. Theory*, 29(5), 656-664, 1983.
- F. G. Leonardi. A generalization of the pst algorithm: modelling the sparse nature of protein sequences. *Bioinformatics*, 22(7), 2006.
- A. Galves, E. Löcherbach. Stochastic chains with memory of variable length. *TICSP Series 38*, 117-133, 2008.

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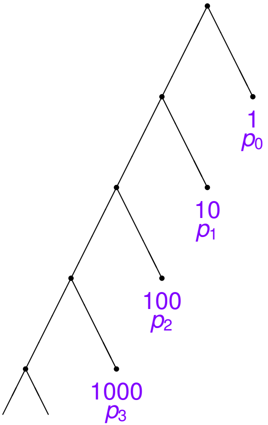
- It's a chain of infinite order.
- Question of the existence of a stationary process consistent with (τ, p).
- The simplest example is the sparse chain.

The sparse chain



The contexts have the form

$\tau = \cup_{k=0}^{+\infty} 10^k$.

The transition probabilities

$p(1|0^k 1) = p_k, k \in \mathbb{N}$.

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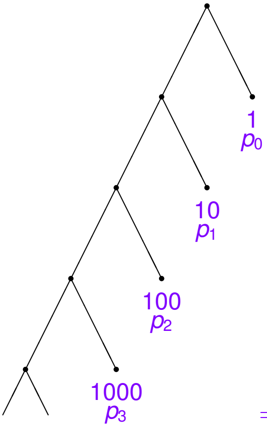

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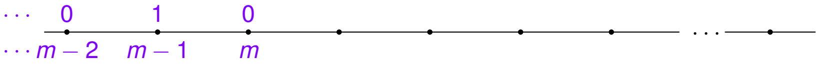
Remark: when a 1 appears, the process "forgets" the past.

$\Longrightarrow$ the instant where a 1 appears are called regeneration times.

The sparse chain is in fact a regeneration process

Suppose given $x_{-\infty}^{m}$



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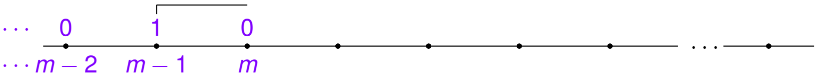

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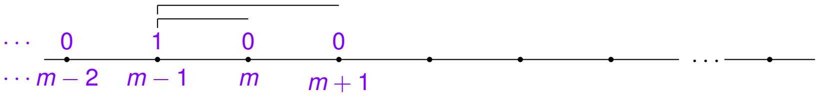

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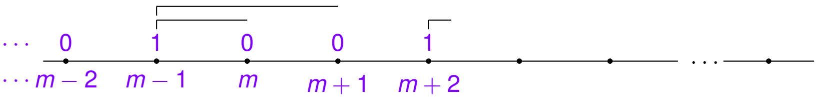

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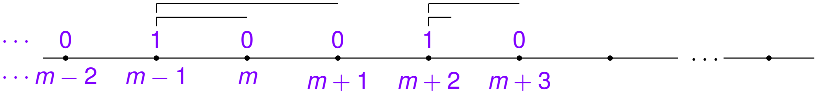

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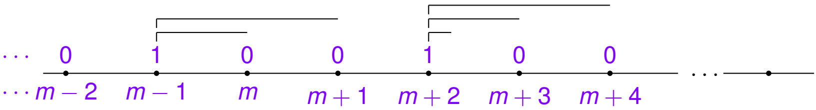

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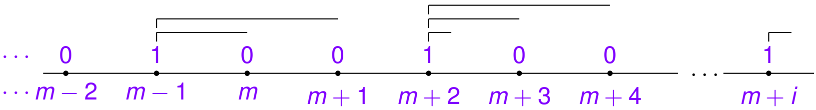

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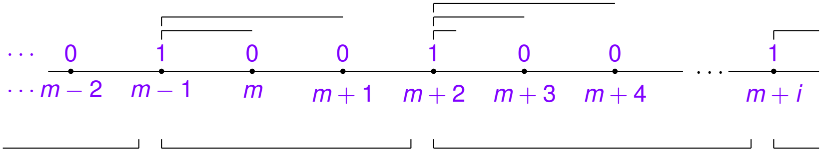
We can decompose the realization in independant blocs, identically distributed.

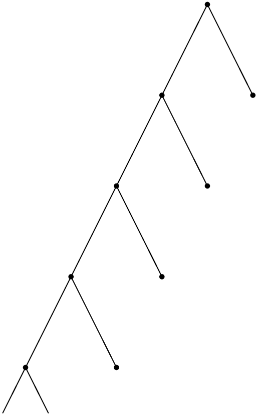


If $\{p_k\}_{k\in\mathbb{N}}$ allows infinitely many 1's, then we have a regeneration scheme.

The problem of the talk

We want to extand this kind of result to more general trees.

The class of trees we are considering



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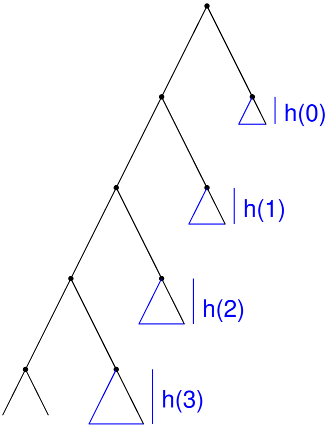

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Let's generalize adding to each context of the form 10^k, a tree $\hat{\tau}^k$ of height $h(k)$:

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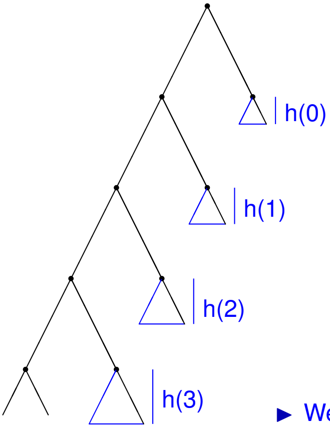

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- We are searching for a condition on $h(k)$ to obtain a regeneration scheme for this new tree.

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then for all $\omega \in \tau$, there exists infinitely many instants

$$\ldots < \sigma_0^\omega \leq 0 < \sigma_1^\omega < \sigma_2^\omega \ldots$$

wich are regeneration times for the realisation of (τ, p) in relation to ω.

Extension to more general trees

This result still hold for more more general form of trees, and I explain this now on the blackboard.

References on regeneration scheme

- F. Comets, R. Fernández, and P. A. Ferrari. Processes with long memory: Regenerative construction and perfect simulation. *Ann. Appl. Probab.* vol. 12 3:921-943, 2002.
  - The regeneration scheme is not visible
- G., Galves (in preparation)
  - The regeneration scheme is visible

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.....except the regularity condition, wich is rather restrictive, but very current in the litterature...

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- minimizes the importance of the notion of continuity rate in the study of infinite order stochastic process,
- deals with a quite big class of trees since $h(k)$ can grow exponencially fast,
- should give a new approach to the bootstrap for infinite memory processes on finite alphabet.

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END...