On a problem of statistical inference in null recurrent diffusions

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introduction: on a statistical problem in null recurrent diffusions

observation: one-dimensional diffusion

$$dX_t = [\vartheta f(X_t) + g(X_t)]\, dt + \sigma dW_t \ , \ \ t \geq 0$$

observed continuously in time, over a long time interval

- ϑ unknown parameter of interest, $\in \mathbb{R}$, to be estimated
- $f(x) := \frac{x}{1+x^2}$ known function s.t. $f(x) \sim \frac{1}{x}$ as $x \to \pm\infty$
- g unknown nuisance function with support in some known compact

recurrence / transience properties: the process X is

- transient if $\vartheta \in (\frac{\sigma^2}{2}, \infty)$, recurrent else
- positive recurrent (ergodic) if $\vartheta \in (-\infty, -\frac{\sigma^2}{2})$

restrict to null recurrent submodel $\Theta := (-\frac{\sigma^2}{2}, +\frac{\sigma^2}{2})$ on which

$$m^{\vartheta,g}(dx) \sim \xi_{\pm}^{\vartheta,g} |x|^{\frac{2\vartheta}{\sigma^2}} dx \quad \text{as } x \to \pm\infty$$

invariant measure of infinite total mass.

statistical questions:

- good estimators for ϑ in the null recurrent model Θ
    - in the parametric model only ($g \equiv 0$)
    - semiparametric (unknown $g \in \mathcal{H}_c$, bd mb fcts with supp in $[-c, c]$)
- speed of convergence, limit laws, efficiency bounds
    - via local asymptotic mixed normality (LAMN)
    - via least favorable one-parametric submodels and local asymptotic minimax bounds

($\longrightarrow$ H. and Kutoyants 2003) probabilistic questions:

need local models, likelihood ratio processes, convergence of score and information; however: basic assumption in martingale limit theorems ($\longrightarrow$ Jacod and Shiryaev 1987)

- sequences of martingales $(M_t^n)_{t\geq 0}$ such that for every t fixed angle bracketts $\langle M^n \rangle_t$ converge *in probability* as $n \to \infty$

is not satisfied: we do have not more than convergence in law

but: in this model very comfortable situation where weak convergence results are available, based on *recurrent cycles* (perhaps artificially associated through a sequence of accompanying Harris processes)

- regular variation: resolvent of the semigroup, tails of invariant measure, norming functions, tails of life cycle length distributions ...
- weak convergence in $\mathbb{D}$: additive functionals, martingales, ...

($\longrightarrow$ H. and Löcherbach 2003)

in general null recurrent models, *weak convergence is not available*, but *tightness rates for additive functionals* do exist ($\longrightarrow$ Loukianova and Loukianov 2008, Löcherbach and Loukianova 2008) type of weak convergence results for estimators under null recurrence:

need *stable increasing process* S^α of index $0 < \alpha < 1$:

- all paths càdlàg nondecreasing, $S^\alpha_0 = 0$
- S^α PIIS with $E(e^{-\lambda S^\alpha_t}) = e^{-t\,\lambda^\alpha}$, $\lambda \geq 0$, $t \geq 0$

and *Mittag-Leffler process* W^α defined as the process inverse of S^α

- $W^\alpha_t \;:=\; \inf\{v : S^\alpha_v > t\}\;,\; t \geq 0$
- all paths continuous and nondecreasing, $W^\alpha = 0$

then for some $\alpha = \alpha(\vartheta)$ in $(0,1)$ and for Brownian motion B subject to *independent* time change $t \to W^\alpha_t$, good estimators for ϑ are such that

$$n^{\alpha/2}\left(\widetilde{\vartheta}_{\cdot n} - \vartheta\right) \;\longrightarrow\; C(\vartheta, g)\,\frac{B(W^\alpha)}{W^\alpha}$$

(weak convergence in $\mathbb{D}$, as $n \to \infty$, under $P_{\vartheta,g}$)

main references for this talk:

- statistical theory of LAMN: LeCam 1968, Hájek 1970, Jeganathan 1982, Davies 1985, LeCam and Yang 1990
  (H.: *Asymptotische Statistik*, in preparation, on my homepage)
- additive functionals in null recurrent Markov processes: Darling and Kac 1957, Bingham and Goldie and Teugels 1987, Khasminskii 1980
- weak convergence for semimartingales: Jacod and Shiryaev 1987
- limit theorems for martingale additive functionals: Greenwood and Resnick 1979, Touati 1988, H. and Löcherbach 2003
- statistical applications: (H. 1990+1993), H. and Kutoyants 2003

limit theorems for null recurrent Markov processes

($\longrightarrow$ Touati 1988, $\longrightarrow$ H. and Löcherbach 2003)

assumption: $X = (X_t)_{t \geq 0}$ càdlàg, strongly Markov, invariant measure μ, Polish state space $(E, \mathcal{E})$, X is recurrent (Harris)

$$A \in \mathcal{E}\,,\ \mu(A) > 0\ :\ \int_0^\infty 1_A(X_s)\, ds = \infty \ \text{ a.s. } (\forall \text{ starting pt } x)$$

where μ has *infinite* total mass: $\mu(E) = \infty$

aim: consider $M = (M_t)_{t \geq 0}$ in $\mathcal{M}^{2,loc}(\mathbb{F}^X)$ with $E_\mu(\langle M \rangle_1) < \infty$

- M is local martingale and additive functional of X
- angle brackett$\langle M \rangle$ is an *integrable* additive functional of X

and prove limit theorems (weak convergence in $\mathbb{D}(\mathbb{R}^+, \mathbb{R} \times \mathbb{R})$) for pairs

$$(\, M_{tn}\,,\ \langle M \rangle_{tn})_{t \geq 0}\ ,\ n \to \infty$$

special case: recurrent atom

assumption: X admits a *recurrent atom* A

- $A \in \mathcal{E}$, $\mu(A) > 0$
- $\sigma_A := \inf\{t : X_t \in A\}$, $\tau_A := \inf\{t : X_t \notin A\}$ are $\mathbb{F}$-stopping times
- $\mathcal{L}(X_{\tau_A} | X_0 = x) =: \rho_A$ does not depend on starting pt x

assumption: there is a *life cycle decomposition* $(R_n)_n$: $R_0 = 0$, for $n \geq 1$

- $R_n < \infty$ a.s., $R_n = R_{n-1} + R_1 \circ \theta_{R_{n-1}}$ a.s.
- $\mathcal{L}(X_{R_n}) = \rho_A$
- $(X_{R_n+s})_{s \geq 0}$ independent of $\mathcal{F}_{R_n^-}$

then: decompose trajectory of X into iid segments

$$X 1_{[[R_n, R_{n+1}[[} \; , \; n \geq 1$$

(plus some initial segment) $\hookrightarrow$ theorem:

theorem 1 : if as $t \to \infty$

$$(*1) \qquad \int_0^t P(R_2 - R_1 > s)\, ds \;\sim\; \frac{1}{\Gamma(2-\alpha)}\, t^{1-\alpha}\, \ell(t)$$

for some function $\ell(\cdot)$ slowly varying at ∞, some $0 < \alpha \leq 1$, then

$$\left(\sqrt{\frac{\ell(n)}{n^\alpha}}\, M_{\cdot n} \,,\, \frac{\ell(n)}{n^\alpha}\, \langle M \rangle_{\cdot n} \right) \longrightarrow \left(J^{1/2} B(W^\alpha) \,,\, J\, W^\alpha \right)$$

(weak convergence in $\mathbb{D}(\mathbb{R}^+, \mathbb{R}\times\mathbb{R})$ as $n \to \infty$) where

$$(*2) \qquad J \;:=\; E\left(\langle M \rangle_{R_2} - \langle M \rangle_{R_1} \right) \;=\; cst \cdot E_\mu\left(\langle M \rangle_1 \right)$$

remarks : i) null recurrent cases with $\alpha = 1$ do exist: here $S^\alpha = id = W^\alpha$ deterministic, as under ergodicity
ii) regular variation $(*1)$ with $0 < \alpha \leq 1$ is a necessary condition for weak convergence to nontrivial continuous limit processes (by Darling and Kac 1957)

scheme of proof: i) $(R_n)_n \hookrightarrow$ iid segments in trajectory of $X \hookrightarrow$ classical sums of iid random variables
ii) convergence of bivariate random walks
and attraction to (B, S^α) (Greenwood and Resnick 1978): here Brownian motion B and stable process S^α are necessarily independent
ii) weak convergence in Skorohod paths spaces $\mathbb{D}$, time change arguments $\square$

general case: recurrent atoms do not exist ...

method used in H. and Löcherbach 2003: we construct X *jointly with an accompanying sequence of processes* $X^m = (X_t^m)_{t \geq 0}$ such that

- except on suitably defined intervals of independent exponential length $[[\tau_k^m, \tau_{k+1}^m[[$, $k \geq 1$, 'more and more sparse' as m grows to ∞, the trajectory of X^m coincides with the trajectory of X
- for every m, X^m is
  - a functional of X with respect to the filtration $\mathbb{F}^X$
  - strongly Markov with respect to its own past, Harris recurrent
  - we can define a recurrent atom and a life cycle decomposition
- for every m, we can apply theorem 1 to the process X^m, with certain $(R_n^m)_n$, and with certain constants depending on m

then as $m \to \infty$, controlling for every m fluctuations of $\mathbb{F}^X$-martingales on the exceptional intervals $[[\tau_k^m, \tau_{k+1}^m[[$, $k \geq 1$, 'more and more sparse' as m grows to ∞, we generalize theorem 1 to general Harris processes X:

theorem 2 : the conclusion of theorem 1

$$\left(\sqrt{\frac{\ell(n)}{n^\alpha}}\, M_{\cdot n}\,,\ \frac{\ell(n)}{n^\alpha}\,\langle M\rangle_{\cdot n} \right) \longrightarrow \left(J^{1/2} B(W^\alpha)\,,\ J\, W^\alpha \right)$$

(weak convergence in $\mathbb{D}(\mathbb{R}^+, \mathbb{R}\times\mathbb{R})$ as $n \to \infty$) holds with

$$J \;=\; E_\mu \left(\langle M\rangle_1 \right)$$

under the following condition on the resolvent of X: as $t \to \infty$

$$(*3) \quad \left(R_{1/t}\, g \right)(x) \;:=\; E_x \left(\int_0^\infty e^{-(1/t)\,v}\, g(X_v)\, dv \right) \;\sim\; \frac{1}{\ell(t)}\, t^\alpha\, \mu(g)$$

for some $g(\cdot)$ nonnegative and $\mathcal{E}$-measurable, and for μ-a.a. $x \in E$.

remark : $(*3) \iff (*1)$ whenever the process has life cycles

more on the accompanying sequence X^m, $m \geq 1$, for X:

i) embed X into a richer Harris structure $\mathbb{X}$:

- prepare iid exponential waiting times σ_i, $i \geq 1$, independent of X
  $\hookrightarrow$ grid of random times $\tau_n \uparrow \infty$ independent of X, $\tau_n := \sum_{i=1}^n \sigma_i$
- discrete-time chain $(X_{\tau_n})_n$ has one-step t.p.

$$U^1(x, dy) \;=\; \int_0^\infty e^{-t}\, P_t(x, dy)\, dt$$

- for some $\alpha > 0$, some 'small' set $C \in \mathcal{E}$ with $\mu(C) > 0$, some probability measure $\nu \sim \mu(\cdot \cap C)$, Nummelin's splitting condition holds:

$$(+) \qquad U^1(x, dy) \;\geq\; \alpha\, 1_C(x)\, \nu(dy)$$

- prepare U_i, V_j iid $\sim \mathcal{R}(0,1)$, independent of X, and define a new Harris process (which contains X as first component) by

$$\mathbb{X} \;:=\; \sum_{n=0}^{\infty} 1_{[[\tau_n, \tau_{n+1}[[}(t)\, (\, X_t, X_{\tau_n}, U_n, V_n \,)$$

- 'identify' X with the new process $\mathbb{X}$

ii) define X^m from X:

- need third components $(U_n)_n$ in $\mathbb{X}$, and set C of splitting condition $(+)$
- on intervals $[[\tau_n, \tau_{n+1}[[$ where $X_{\tau_n} \in C$ and $U_n < 2^{-m}$, *freeze trajectory*

$$\mathbb{X}\, 1_{[[\tau_n,\tau_{n+1}[[} \;=\; \left((\, X_t, X_{\tau_n}, U_n, V_n \,)\, 1_{[[\tau_n,\tau_{n+1}[[}(t) \,\right)_{t\geq 0}$$

  to its value at time τ_n

$$\mathbb{X}^m\, 1_{[[\tau_n,\tau_{n+1}[[} \;:=\; \left((\, X_{\tau_n}, X_{\tau_n}, U_n, V_n \,)\, 1_{[[\tau_n,\tau_{n+1}[[}(t) \,\right)_{t\geq 0}$$

- on all other intervals $[[\tau_n, \tau_{n+1}[[$, do nothing:

$$\mathbb{X}^m\, 1_{[[\tau_n,\tau_{n+1}[[} \;:=\; \mathbb{X}\, 1_{[[\tau_n,\tau_{n+1}[[}$$

- write $[[\tau_k^m, \tau_{k+1}^m[[$ for the kth interval on which $\mathbb{X}^m$ is different from $\mathbb{X}$

(whenever possible, write for short X^m for $\mathbb{X}^m$, $m \geq 1$)

call freezing intervals $[[\tau_k^m, \tau_{k+1}^m[[$, $k \geq 1$, $m \to \infty$, *exceptional* since

- with increasing m, freezing becomes more and more sparse
- a given $[[\tau_k^m, \tau_{k+1}^m[[$, freezing interval in the process $\mathbb{X}^m$, will remain a freezing interval for the process $\mathbb{X}^{m+1}$ w.pr. $\frac{1}{2}$

view $\mathbb{X}^m$ as functional of $\mathbb{X}$: trajectories of $\mathbb{X}^m$ approach those of $\mathbb{X}$ as $m \to \infty$

iii) $\mathbb{X}^m$ is a Harris process in its own right:

- in $\mathbb{X}^m$, random times τ_n, $n \geq 1$, occur at constant rate 1
- in $\mathbb{X}^m$, at the end of an exceptional interval $[[\tau_k^m, \tau_{k+1}^m[[$, successor state for X_{τ_n} is selected according to $U^1(X_{\tau_n}, \cdot)$
- with respect to its own past, $\mathbb{X}^m$ is strongly Markov, is recurrent (Harris), and visits $C \times C \times (0, 2^{-m}) \times (0, 1)$ infinitely often on the time grid τ_n, $n \geq 1$

iv) in the Harris process $\mathbb{X}^m$, $\exists$ recurrent atom and life cycle decomposition:

need forth components $(V_n)_n$ of $\mathbb{X}^m$, and C, α, ν of splitting condition $(+)$

on all *exceptional intervals*

$$[[\tau_k^m, \tau_{k+1}^m[[=: [[\tau_\ell, \tau_{\ell+1}[[\ , \ \ k \geq 1$$

we have independent exponential clock running up to time $\tau_{\ell+1}$, have X_{τ_ℓ} as current value, and do Nummelin splitting ($\longrightarrow$ Nummelin 1978) at time $\tau_{\ell+1-}$:

- if $V_\ell < \alpha$, select $X_{\tau_{\ell+1}}$ according to $\nu(dy)$
- else, select $X_{\tau_{\ell+1}}$ according to the law $\frac{1}{1-\alpha}(U^1(X_{\tau_\ell}, dy) - \alpha\nu(dy))$

thus $A = C \times C \times (0, 2^{-m}) \times (0, \alpha)$ becomes a recurrent atom, and

- $\{V_\ell < \alpha\}$ at time $\tau_{\ell+1-}$ prescribes beginning of a new life cycle at $\tau_{\ell+1}$
- at such $\tau_{\ell+1}$, the starting value $X_{\tau_{\ell+1}} \sim \nu(dy)$ is independent of $\mathcal{F}_{\tau_{\ell+1-}}$

$\hookrightarrow$ trajectory of $\mathbb{X}^m$ decomposed into iid segments (plus initial segment) □