Localization error estimates for HJB equations

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Given a stochastic differential equation

$$dX(s) = b(s, X(s), \pi(s))ds + \sigma(s, X(s), \pi(s))dW(s), \quad t \leq s \leq T$$

whose solution is $X(s) = X_s^{t,x}$ with initial data $X(t) = x$.
Consider the stochastic control problem

$$V(t,x) = \sup_{\pi \in \mathcal{A}} \mathbb{E} \left\{ \int_t^T L(s, X_s^{t,x}, \pi(s))ds + \psi(X_T^{t,x}) \right\}.$$

$V(t,x)$ is the classical or viscosity solution to the *Hamilton-Jacobi-Bellman* (HJB) PDE:

$$\begin{cases} \dfrac{\partial V}{\partial t} + \mathcal{H}(t,x,D_x V, D_x^2 V) = 0, \quad (t,x) \in [0,T) \times \mathbb{R}^n, \\ \\ V(T,x) = \psi(x), \quad x \in \mathbb{R}^n, \end{cases}$$

where

$$\mathcal{H}(t,x,p,A) = \sup_{u \in \Pi} \left[b \cdot p + \frac{1}{2} Tr(A\sigma\sigma^T) + L \right].$$

Numerical approximation of V:

$$\begin{cases} \dfrac{V^h(t,x) - V^h(t-h,x)}{h} + \bar{\mathcal{H}}^h = 0, \quad (t,x) \in [0,T) \times O, \\ \\ V^h(t,x) = \Psi(t,x), \quad (t,x) \in ([0,T) \times \partial O) \bigcup (\{T\} \times \overline{O}). \end{cases}$$

Questions

- What is a suitable $\Psi(t,x)$? Sure we can take $\Psi(T,x) = \psi(x)$, but how about Ψ on $([0,T) \times \partial O)$?
- Given $\Psi(t,x)$, especially $\Psi(t,x) = \psi(x)$, what is the error between V and V^h?

V^h corresponds to the HJB equation

$$\begin{cases} \dfrac{\partial \widetilde{V}}{\partial t} + \mathcal{H}(t, x, D_x \widetilde{V}, D_x^2 \widetilde{V}) = 0, \quad (t, x) \in [0, T) \times O, \\ \widetilde{V}(t, x) = \Psi(t, x), \quad (t, x) \in ([0, T) \times \partial O) \bigcup (\{T\} \times \overline{O}), \end{cases}$$

and then in some sense (classical or viscosity solution) is related to

$$\widetilde{V}(t, x) = \sup_{\pi \in \mathcal{A}} \mathbb{E} \left\{ \int_t^{\tau \wedge T} L(s, X_s^{t,x}, \pi(s)) ds + \Psi(\tau \wedge T, X_{\tau \wedge T}^{t,x}) \right\},$$

where $\tau = \inf\{s : s \geq t, X_s^{t,x} \notin O\}$ is the exit time of $X_s^{t,x}$ from O.

$V - V^h = ?$

$$V : \frac{\partial V}{\partial t} + \mathcal{H} = 0, \ \mathbb{R}^n \qquad V = \sup \mathbb{E}\left\{ \int_t^T + \psi \right\}$$

$$V^h : \frac{\Delta V^h}{h} + \bar{\mathcal{H}}^h = 0, \ O \qquad \widetilde{V} = \sup \mathbb{E}\left\{ \int_t^{\tau \wedge T} + \Psi \right\}$$

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$$\widetilde{V} : \frac{\partial \widetilde{V}}{\partial t} + \mathcal{H} = 0, \; O$$

Instead of $V - V^h$, we consider $V - \widetilde{V} = \sup \mathbb{E}(\cdot) - \sup \mathbb{E}(\cdot)$.

Theorem

$$|V(t,x)-\widetilde{V}(t,x)| \le \sup_{0\le s\le T,\, y\in\partial O} |V(s,y)-\widetilde{V}(s,y)| \sup_{\pi\in\mathcal{A}} \mathbb{P}(\tau \le T).$$

Proof.

Use the dynamic programming principle:

$$V(t,x) = \sup_{\pi\in\mathcal{A}} \mathbb{E}\left\{ \int_t^{\theta\wedge T} L(s, X_s^{t,x}, \pi(s))ds + V(\theta\wedge T, X_{\theta\wedge T}^{t,x}) \right\},$$

where $\theta \ge t$ is a stopping time. □

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A financial market model

“Bond”: one riskless asset

$$\frac{dB(s)}{B(s)} = r(s)ds, \quad t \le s \le T.$$

“Stocks”: *n* risky assets

$$\frac{dP_i(s)}{P_i(s)} = \mu_i(s) + \sum_{j=1}^{n} \sigma_{ij}(s)dW_j(s), \quad t \le s \le T.$$

A financial market model

"Bond": one riskless asset

"Stocks": n risky assets

Wealth process

$$dX(s) = (X(s) - \sum_{i=1}^{n} \pi_i(s)X(s))\frac{dB(s)}{B(s)} + \sum_{i=1}^{n} \pi_i(s)X(s)\frac{dP_i(s)}{P_i(s)},$$

where $\pi_i(s)$ is the proportion of wealth invested in the i^{th} stock.

In a complete market

Theorem

$$|V(t,x) - \widetilde{V}(t,x)| \leq \sup_{0 \leq s \leq T} |V(s,\beta) - \widetilde{V}(s,\beta)| F(\beta)$$

for $t \in [0,T]$, $x \in (0,\beta)$. Especially, when r, μ_i and σ_{ij} are constants, for $t \in [0,T]$, $x \in (0, \beta e^{-r(T-t)})$,

$$F(\beta) = \Phi\left(\Phi^{-1}\left(\frac{x}{\beta} e^{r(T-t)}\right) + |\theta|\sqrt{T-t}\right),$$

where θ is a known constant and $\Phi(\cdot)$ is the cumulative normal distribution function.

Lemma

Given $O = (0, \beta)$.

$$\sup_{\pi \in \mathcal{A}} \mathbb{P}(\tau \leq T) = \sup_{\pi \in \mathcal{A}} \mathbb{P}(X_T^{t,x} \geq \beta).$$

Proof.

1. Since $X_s^{t,x}$ is positive, $\tau = \inf\{s : s \geq t, X_s^{t,x} \geq \beta\}$.
2. $\sup_\pi \mathbb{P}(\tau \leq T) \geq \sup_\pi \mathbb{P}(X_T^{t,x} \geq \beta)$. Trivial.
3. Imagine an investment strategy: invest all the money in the riskless asset once the wealth attains β. Then the terminal wealth will be greater than β.
4. Difficulty: to prove this strategy is progressively measurable.

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Calculate $\sup_{\pi\in\mathcal{A}} \mathbb{P}(X_T^{t,x} \geq \beta)$

"To maximize the probability that the portfolio reaches a given target", this problem has been studied by several researchers. See Spivak and Cvitanić [1], Browne [2].

Error estimates in high-dimensional space

Given a $\mathbb{R}^n$-valued state process $X(s)$ evolving as follows

$$dX(s) = b(s, X(s), \pi(s))ds + \sigma(s, X(s), \pi(s))dW(s), \quad t \leq s \leq T.$$

Denote this process by $X_s^{t,x}$ if $X(t) = x$. And assume

$$|b(t,x,u) - b(t,y,u)| + \|\sigma(t,x,u) - \sigma(t,y,u)\| \leq K|x-y|,$$

$$|b(t,x,u)| + \|\sigma(t,x,u)\| \leq K(1+|x|),$$

for some constant $K > 1$.

Error estimates in high-dimensional space

Theorem

Given $O = \{x | x \in \mathbb{R}^n, |x| < R\}$ *for* $R > 0$*.* τ *be the exit time of* $X_s^{t,x}$ *from O. Assume*

$$\ln(1 + R^2) - \ln(1 + |x|^2) - 9K^2(T - t) > 0.$$

We have

$$\mathbb{P}(\tau \leq T) \leq 2e^{-\frac{9}{2}K^2(T-t)} \left(\frac{1 + |x|^2}{1 + R^2}\right)^{\frac{1}{18K^2(T-t)} \ln \frac{1+R^2}{1+|x|^2} - 1}.$$

Error estimates in high-dimensional space

Proof.

1. Recall the exponential inequality for local martingales: Let $\{M_t,\ t \in [0, T]\}$ be a continuous local martingale. For any $\delta < 0$ and $\rho > 0$

$$\mathbb{P}\Big\{\langle M\rangle_T < \rho,\ \sup_{0\le t\le T} |M_t| \ge \delta\Big\} \le 2\exp\Big(-\frac{\delta^2}{2\rho}\Big).$$

2. $Z(s) = \ln(1 + |X_s^{t,x}|^2) = A_s + M_s$, where M_s is a local martingale with a bounded quadratic variation and A_s is also bounded.
3. $\tau \le T \Leftrightarrow \sup_{t\le s\le T} |Z(s)| \ge \ln(1 + R^2)$.

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Outline

We are developing a numerical procedure to approximate the boundary conditions.

- This procedure is based on the Robbins-Monro algorithm. Rate of convergence (in progress).
- Rate of convergence in the Martingale CLT (in progress).

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