Exact Monte Carlo methods
Application : Pricing of continuous Asian options

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In many application, especially finance, we are faced with the problem of computing

$$C_0 = \mathbb{E}(f(X_T))$$

where f is a given function X is the solution of a one-dimensional SDE :

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t; \quad X_0 = x$$

Monte Carlo methods for doing the job in case $\mathcal{L}(X_T)$ is unknown :

- Approximate X_T by a discretized process (Euler, Milstein ...).
- Do rejection sampling (exact algorithm of Beskos and al. [1]).
- Construct unbiased estimators, ideally simple to simulate from and with low variance.

Advantage of an exact method : no discretization bias

Outline

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Assume that we can find a simulable process Z such that, by a change of measure,

$$C_0 = \mathbb{E}\left(f(Z_T)\psi(Z_T)\exp\left[-\int_0^T \phi(Z_t)dt\right]\right) \tag{1}$$

where ψ and ϕ are two explicit functions.

Early nineties : Wagner [18] constructs several unbiased estimators of such an expectation by expanding the exponential term in a power series.

Recently : Beskos and al. [2] and Fearnhead and al. [5] extend his idea to get the Poisson estimator and the generalized Poisson estimator.

The unbiased estimator (U.E)

Suppose that,

- conditionnaly on $Z = (Z_t)_{t\in[0,T]}$,
  - $N \sim p_Z$ with p_Z a positive probability measure on $\mathbb{N}$.
  - $(V_i)_{i\in\mathbb{N}^*} \overset{\text{i.i.d}}{\sim} q_Z$ with q_Z a positive probability density on $[0,T]$.
  - N and $(V_i)_{i\in\mathbb{N}^*}$ are independent and $c_Z \in \mathbb{R}$.
- (I.C) $\mathbb{E}\left(|f(Z_T)\psi(Z_T)|e^{-c_Z T}\exp\left[\int_0^T |c_Z - \phi(Z_t)|dt\right]\right) < \infty$

Lemma

$f(Z_T)\psi(Z_T)e^{-c_Z T}\dfrac{1}{p_Z(N)\,N!}\prod_{i=1}^{N}\dfrac{c_Z - \phi(Z_{V_i})}{q_Z(V_i)}$ *is an unbiased estimator of* C_0.

Proof :

$$\begin{aligned}\Delta(Z) &:= \mathbb{E}\left(f(Z_T)\psi(Z_T)e^{-c_Z T}\frac{1}{p_Z(N)\,N!}\prod_{i=1}^{N}\frac{c_Z-\phi(Z_{V_i})}{q_Z(V_i)}\,\Big|\,Z\right)\\ &= f(Z_T)\psi(Z_T)e^{-c_Z T}\sum_{n=0}^{+\infty}\frac{\left(\int_0^T c_Z-\phi(Z_t)dt\right)^n}{p_Z(n)\,n!}p_Z(n)\\ &= f(Z_T)\psi(Z_T)\exp\left(-\int_0^T\phi(Z_t)dt\right).\end{aligned}$$

□

So, we can compute C_0 by a simple Monte Carlo :

$$C_0 \approx \frac{1}{n}\sum_{i=1}^{n} f(Z_T^i)\psi(Z_T^i)e^{-c_Z T}\frac{1}{p_Z(N^i)\,N^i!}\prod_{j=1}^{N^i}\frac{c_Z-\phi(Z^i_{V^i_j})}{q_Z(V^i_j)}.$$

Levers for variance reduction

- Importance sampling : the choice of the process Z is decisive.
- The choice of the parameters p_Z and q_Z.

  A simple case : $e^{\int_0^T g(t)dt} = \mathbb{E}\left(\frac{1}{p(N)\,N!}\prod_{i=1}^N \frac{g(V_i)}{q(V_i)}\right)$ where $g : [0,T] \to \mathbb{R}$.

$$q_{opt}(t) = \frac{|g(t)|}{\int_0^T |g(t)|dt}\,\mathbb{1}_{[0,T]}(t) \text{ and } p_{opt}(n) = \frac{\left(\int_0^T |g(t)|dt\right)^n}{n!}\,e^{-\int_0^T |g(t)|dt}$$

- Shifting : the parameter c_Z.

How to obtain such a process Z ? (Beskos and al. [1])

- Without loss of generality, suppose that X is solution of

$$\begin{cases} dX_t &= a(X_t)dt + dW_t \\ X_0 &= x. \end{cases} \quad (2)$$

$\big($if X has a non-constant diffusion coefficient $\sigma(.)$, make the change of variables $Y_t = \eta(X_t)$ with $\eta(x) = \int_.^x \frac{1}{\sigma(u)} du$ $\big)$

- Denote by $(W_t^x)_{t \in [0,T]}$ the process $(W_t + x)_{t \in [0,T]}$.

Assumption 1 :

$$L_t = \exp\left[\int_0^t a(W_u^x) dW_u^x - \frac{1}{2}\int_0^t a^2(W_u^x) du\right] \textit{ is a martingale.}$$

How to obtain such a process Z ? (Beskos and al. [1])

- *Assumption 2 : a* is continuously differentiable.

  Denote by A the primitive of the drift a. By Itô's lemma, one gets

$$L_T = \exp\left[A(W_T^x) - A(x) - \frac{1}{2}\int_0^T a^2(W_t^x) + a'(W_t^x)dt\right].$$

- Consider the process $(Z_t)_{t\in[0,T]}$ distributed according to

$$\mathbb{Q}_Z = \int_{\mathbb{R}} \mathcal{L}\Big((W_t^x)_{t\in[0,T]}|W_T^x = y\Big)\rho(y)dy.$$

  where ρ is a positive density on the real line (another lever for variance reduction). By the Girsanov theorem,

$$C_0 = \mathbb{E}\left(f(Z_T)\psi(Z_T)\exp\left[-\int_0^T \phi(Z_t)dt\right]\right) \quad (3)$$

  where $\psi : z \mapsto \frac{e^{A(z)-A(x)-\frac{(z-x)^2}{2T}}}{\sqrt{2\pi}\rho(z)}$ and $\phi : z \mapsto \frac{a^2(z)+a'(z)}{2}$.

Outline

- In the Black & Scholes framework, under the risk-neutral measure

$$\frac{dS_t}{S_t} = (r - \delta)dt + \sigma dW_t$$

so $S_t = S_0 e^{\sigma W_t + \gamma t}$ where $\gamma = r - \delta - \frac{\sigma^2}{2}$.

- Price of a continuous Asian option with pay-off f :

$$C_0 = \mathbb{E}\left(e^{-rT} f\left(\frac{1}{T}\int_0^T S_u\right)\right)$$

$\Rightarrow$ No simple closed form solution

- Numerical methods :
  - Analytical approximations (Turnbull and Wakeman [15], Vorst [17], Levy [11] and more recently Lord [12]).
  - PDE methods (see Vecer [16], Rogers and Shi [13], Ingersoll [8], Lelievre and Dubois [4]).
  - Monte Carlo simulation methods (see Kemna and Vorst [9], Broadie and Glasserman [3], Fu and al. [6], Lapeyre and Temam [10]).
  - Laplace transform inversion methods (see Geman and Yor [7]).

- A priori, a two dimensional problem : let $\overline{S}_t = \frac{1}{t}\int_0^t S_u du$, then

$$\begin{cases} dS_t &= S_t((r-\delta)dt + \sigma dW_t) \\ d\overline{S}_t &= (-\frac{1}{t}\overline{S}_t + \frac{S_t}{t})dt \end{cases}$$

But, with a suitable change of variables (Rogers and Shi [13]), we can reduce the dimension to one :

$$\xi_t = S_0 \int_0^t e^{\sigma(W_t - W_u) + \gamma(t-u)} du$$

$\xi_T = \int_0^T S_0 e^{\sigma(W_T - W_{T-s}) + \gamma s} ds$ and $\displaystyle\int_0^T S_u du$ have the same law so

$$C_0 = \mathbb{E}\left(e^{-rT} f(\tfrac{1}{T}\xi_T)\right)$$

where $(\xi_t)_{t\in[0,T]}$ is solution of

$$\begin{cases} d\xi_t &= S_0 dt + \xi_t\left(\sigma dW_t + (\gamma + \frac{\sigma^2}{2})dt\right) \\ \xi_0 &= 0 \end{cases}$$

- Need for a new change of variables :

$$\xi_t = \frac{1}{t} S_0 \int_0^t e^{\sigma(W_t - W_u) + \gamma(t-u)} du$$

$\xi_T = \frac{1}{T} \int_0^T S_0 e^{\sigma(W_T - W_{T-s}) + \gamma s} ds$ and $\frac{1}{T} \int_0^T S_u du$ have the same law so

$$C_0 = \mathbb{E}\left(e^{-rT} f(\xi_T)\right)$$

where $(\xi_t)_{t \in [0,T]}$ is solution of

$$\begin{cases} d\xi_t &= \frac{\xi_0 - \xi_t}{t} dt + \xi_t \left(\sigma dW_t + (\gamma + \frac{\sigma^2}{2}) dt\right) \\ \xi_0 &= S_0 \end{cases}$$

$$X_t = \log(\frac{\xi_t}{\xi_0}) \Rightarrow \begin{cases} dX_t &= \sigma dW_t + \gamma dt + \frac{e^{-X_t}-1}{t} dt \\ X_0 &= 0. \end{cases} \qquad (4)$$

Difficulty : singularity of the drift term for $t \to 0$ prevents X from having an a.c law w.r.t the law of W.

$$X_t = \log(\frac{\xi_t}{\xi_0}) \Rightarrow \begin{cases} dX_t = \sigma dW_t + \gamma dt + \frac{e^{-X_t}-1}{t} dt \\ X_0 = 0. \end{cases} \quad (4)$$

Difficulty : singularity of the drift term for $t \to 0$ prevents X from having an a.c law w.r.t the law of W.

Consider instead

$$dZ_t = \sigma dW_t + \gamma dt - \frac{Z_t}{t} dt;\ Z_0 = X_0 = 0. \quad (5)$$

Lemma

Existence and strong uniqueness hold for (4) and (5). Moreover,

$$Z_t = \frac{\sigma}{t} \int_0^t s\, dW_s + \frac{\gamma}{2} t \text{ is a solution of (5).}$$

$(Z_t)_{t \in [0,T]}$ is a Gaussian process and $Z_T \sim \mathcal{N}(\frac{\gamma}{2} T, \frac{\sigma^2}{3} T)$.

Proposition

$$L_t = \exp\left[\int_0^t \frac{e^{-Z_s} - 1 + Z_s}{\sigma s} dW_s - \frac{1}{2}\int_0^t \left(\frac{e^{-Z_s} - 1 + Z_s}{\sigma s}\right)^2 ds\right]$$

is a martingale and hence $C_0 = \mathbb{E}\left(e^{-rT} f(S_0 e^{Z_T}) L_T\right)$

Proof : By the L.I.L of the Brownian motion, we show that $\forall \epsilon > 0$, there exists a (random) neighborhood of $t = 0$ for which

$$|Z_t| \leq c t^{\frac{1}{2} - \epsilon} \text{ and } |X_t| \leq c t^{\frac{1}{2} - \epsilon}$$

hence, almost surely,

$$\int_0^t \left(\frac{e^{-Z_s} - 1 + Z_s}{\sigma s}\right)^2 ds < \infty \text{ and } \int_0^t \left(\frac{e^{-X_s} - 1 + X_s}{\sigma s}\right)^2 ds < \infty.$$

Existence and strong uniqueness of the SDEs (4) and (5) permits to conclude (see Rydberg [14]). □

Let $A(t,z) = \dfrac{1 - z + \frac{z^2}{2} - e^{-z}}{\sigma^2 t}$. By Itô's lemma

$$A(T,Z_T) = \int_0^T \frac{e^{-Z_t} - 1 + Z_t}{\sigma^2 t} dZ_t - \int_0^T \frac{1 - Z_t + \frac{Z_t^2}{2} - e^{-Z_t}}{\sigma^2 t^2} dt + \int_0^T \frac{1 - e^{-Z_t}}{2t} dt.$$

Finally, $C_0 = \mathbb{E}\left(e^{-rT} f(S_0 e^{Z_T}) e^{A(T,Z_T)} \exp\left[\int_0^T \phi(t,Z_t) dt\right]\right)$ with

$$\phi(t,z) = \frac{1 - z + \frac{z^2}{2} - e^{-z}}{\sigma^2 t^2} - \frac{1 - e^{-z}}{2t} - \frac{e^{-z} - 1 + z}{\sigma^2 t}\left(\frac{e^{-z} - 1 - z}{2t} + \gamma\right).$$

A first conjecture

In order to be able to deal with both calls and puts, we need the following integrability condition (I.C)

Conjecture

$$\mathbb{E}\left(e^{A(T,Z_T)-rT}(e^{Z_T}+1)e^{\int_0^T|\phi(t,Z_t)|dt}\right)<\infty.$$

For a call, this implies that

$$C_0=\mathbb{E}\left(e^{A(T,Z_T)-rT}(S_0e^{Z_T}-K)_+\frac{1}{p(N)\,N!}\prod_{i=1}^{N}\frac{\phi(U_i,Z_{U_i})}{q(U_i)}\right)$$

with well chosen probability distributions p and q.

Choice of the distributions p and q

We need square integrability in order to construct confidence intervals.

$$\mathbb{E}\left(e^{2A(T,Z_T)-2rT} f^2(S_0 e^{Z_T}) \frac{\left(\int_0^T \frac{\phi^2(t,Z_t)}{q(t)} dt\right)^N}{p(N)^2\,(N!)^2} \right) < \infty\,?$$

False for the naive choice of a uniform distribution for q :

Lemma

$\forall \epsilon > 0$, we have a.s. $\phi(t,Z_t) - \frac{2Z_t^3}{3\sigma^2 t^2} + \frac{Z_t}{2t} = \mathcal{O}(t^{-\epsilon})$.

Therefore, we have

$$\int_0^T \frac{\phi^2(t,Z_t)}{t^a} dt < \infty \text{ a.s. if and only if } a < 0.$$

Variance reduction

With $p = \mathcal{P}(c_P T)$ and $q(t) = \frac{1}{2\sqrt{t}\sqrt{T}}\mathbb{1}_{[0,T]}(t)$ (since $\phi \sim \frac{1}{\sqrt{t}}$ near 0), our estimator writes

$$\delta = \frac{1}{m}\sum_{j=1}^{m} e^{A(T,Z_T^j)-rT}(S_0 e^{Z_T^j} - K)_+ \, e^{c_p T - c_{Z^j} T} \prod_{i=1}^{N^j} \frac{2\sqrt{U_i^j}\left(c_{Z^j} - \phi(U_i^j, Z^j_{U_i^j})\right)}{c_p\sqrt{T}}$$

- **Conditioning** : for every simulated trajectory Z^j, we compute

  $$\frac{1}{n}\sum_{k=1}^{n}\prod_{i=1}^{N_k^j} \frac{2\sqrt{U_{i,k}^j}\left(c_{Z^j} - \phi(U_{i,k}^j, Z^j_{U_{i,k}^j})\right)}{c_p\sqrt{T}}$$

  instead of

  $$\prod_{i=1}^{N^j} \frac{2\sqrt{U_i^j}\left(c_{Z^j} - \phi(U_i^j, Z^j_{U_i^j})\right)}{c_p\sqrt{T}}$$

- **Control variate** : we can use $e^{-rT}\left(S_0 e^{Z_T} - K\right)_+$ as a control variate since $Z_T \sim \mathcal{N}(\frac{\gamma}{2}T, \frac{\sigma^2 T}{3})$.

Outline

Numerical test of the conjectures

$$\frac{1}{m}\sum_{j=1}^{m} e^{A(T,Z_T^j)-rT}\left(S_0 e^{Z_T^j}+1\right) e^{c_p T}\prod_{i=1}^{N^j} 2\sqrt{U_i^j}\,\frac{|\phi(U_i^j, Z_{U_i^j}^j)|}{c_p\sqrt{T}} \qquad (6)$$

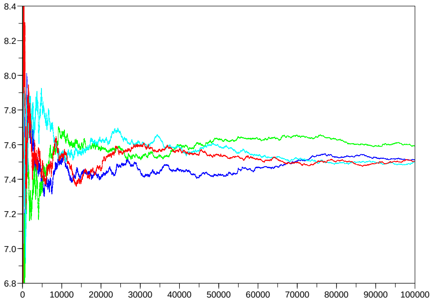

FIG.: Evaluation of (6) with respect to the number m of simulations

Numerical test of the conjectures

$$\frac{1}{m}\sum_{j=1}^{m} e^{2A(T,Z_T^j)-2rT}\left(S_0 e^{Z_T^j}+1\right)^2 e^{2c_pT}\prod_{i=1}^{N^j} 4U_i^j \,\frac{\phi^2(U_i^j, Z^j{}_{U_i^j})}{c_p^2 T} \tag{7}$$

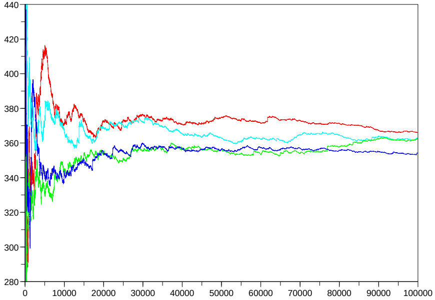

FIG.: Evaluation of (7) with respect to the number m of simulations

Comparison with a standard Monte Carlo method

Set of parameters :

$-S_0 = K = 100$

$-r = 0.1$

$-\sigma = 0.2$

$-T = 1$

Method	Price	L.C.I at 95%	N	CPU
E.C.E std	7.035	0.035	2.10^5	$\sim$1s
E.C.E opt	7.043	0.005	2.10^5	$\sim$1s
MC std (Trap)	7.051	0.053	10^5	$\sim$1s
MC opt (Trap+KV)	7.041	0.002	10^5	$\sim$1s

TAB.: Asian call price with different MC methods. For the standard E.C.E (without variance reduction), we took $c_p = 1$. For the optimized E.C.E, we took $c_p = c_T = \frac{1}{2T}$ and $n = 5$. For MC, the number of time steps is 20.

Comparison with a standard Monte Carlo method

Set of parameters :

$-S_0 = K = 100$

$-r = 0.1$

$-\sigma = 0.2$

$-T = 1$

Method	Price	L.C.I at 95%	N	CPU
E.C.E opt	7.0404	0.0001	7.10^6	~15s
MC opt (Trap+KV)	7.0401	0.0008	10^6	~15s

TAB.: Asian call price with different MC methods. For the optimized E.C.E, we took $c_p = c_T = \frac{1}{2T}$ and $n = 5$. For Trap+KV, the number of time steps is 100.

Outline

Conclusion

Pros

- A MC price of an Asian option that is not prone to discretization bias.

  $\hookrightarrow$ A reliable benchmark

- A competitive method if high precision is required.
- A competitive MC method for pricing Asian like options with pay-off $\alpha S_T + \beta \int_0^T S_u du$, $\alpha \neq 0$.

Cons

- Less competitive than an optimized Monte Carlo (Lapeyre and Temam [10]) for usual precision levels (any other possible variance reduction method ?).
- No theoretical justification of the integrability conjectures.

Merci !

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