Exact Monte Carlo methods
Application: Pricing of continuous Asian options

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In many applications, especially in finance, we are faced with the problem of computing

\[ C_0 = \mathbb{E}(f(X_T)) \]

where \( f \) is a given function, \( X \) is the solution of a one-dimensional SDE:

\[ dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t; \quad X_0 = x \]

Monte Carlo methods for doing the job in case \( \mathcal{L}(X_T) \) is unknown:

- Approximate \( X_T \) by a discretized process (Euler, Milstein …).
- Do rejection sampling (exact algorithm of Beskos and al. [1]).
- Construct unbiased estimators, ideally simple to simulate from and with low variance.

**Advantage of an exact method: no discretization bias**
Outline

1. Exact computation of expectations

2. Application: the pricing of continuous Asian options

3. Numerical results

4. Conclusion
Outline

1. Exact computation of expectations
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Assume that we can find a simulable process $Z$ such that, by a change of measure,

$$C_0 = \mathbb{E}\left(f(Z_T)\psi(Z_T)\exp\left[-\int_0^T \phi(Z_t)dt\right]\right)$$  \hspace{1cm} (1)

where $\psi$ and $\phi$ are two explicit functions.

Early nineties: Wagner [18] constructs several unbiased estimators of such an expectation by expanding the exponential term in a power series.

Recently: Beskos and al. [2] and Fearnhead and al. [5] extend his idea to get the Poisson estimator and the generalized Poisson estimator.
The unbiased estimator (U.E)

Suppose that,

- conditionnally on $Z = (Z_t)_{t \in [0,T]}$,
  - $N \sim p_Z$ with $p_Z$ a positive probability measure on $\mathbb{N}$.
  - $(V_i)_{i \in \mathbb{N}^*} \overset{i.i.d.}{\sim} q_Z$ with $q_Z$ a positive probability density on $[0, T]$.
  - $N$ and $(V_i)_{i \in \mathbb{N}^*}$ are independent and $c_Z \in \mathbb{R}$.

- (I.C) $\mathbb{E}\left( |f(Z_T)\psi(Z_T)| e^{-c_Z T} \exp \left[ \int_0^T |c_Z - \phi(Z_t)| dt \right] \right) < \infty$

**Lemma**

$$f(Z_T)\psi(Z_T)e^{-c_Z T} \frac{1}{p_Z(N) N!} \prod_{i=1}^{N} \frac{c_Z - \phi(Z_{V_i})}{q_Z(V_i)}$$ is an unbiased estimator of $C_0$. 
Proof:

\[
\Delta(Z) := \mathbb{E} \left( f(Z_T)\psi(Z_T)e^{-czT} \frac{1}{p_Z(N) N!} \prod_{i=1}^{N} cZ - \phi(Z_{V_i}) \mid Z \right)
\]

\[
= f(Z_T)\psi(Z_T)e^{-czT} \sum_{n=0}^{+\infty} \frac{\left( \int_0^T cZ - \phi(Z_t)dt \right)^n}{p_Z(n) n!} p_Z(n)
\]

\[
= f(Z_T)\psi(Z_T) \exp \left( - \int_0^T \phi(Z_t)dt \right) .
\]

So, we can compute \( C_0 \) by a simple Monte Carlo:

\[
C_0 \approx \frac{1}{n} \sum_{i=1}^{n} f(Z_T^i)\psi(Z_T^i)e^{-czT} \frac{1}{p_Z(N^i) N^i!} \prod_{j=1}^{N^i} cZ - \phi(Z_{V_{ij}}^i) \frac{1}{q_Z(V_j^i)} .
\]
Levers for variance reduction

- Importance sampling: the choice of the process $Z$ is decisive.
- The choice of the parameters $p_Z$ and $q_Z$.

A simple case: $e^{\int_0^T g(t) dt} = \mathbb{E} \left( \frac{1}{p(N) N!} \prod_{i=1}^N \frac{g(V_i)}{q(V_i)} \right)$ where $g : [0, T] \rightarrow \mathbb{R}$.

$$q_{opt}(t) = \frac{|g(t)|}{\int_0^T |g(t)| dt} \mathbb{1}_{[0,T]}(t) \text{ and } p_{opt}(n) = \frac{\left( \int_0^T |g(t)| dt \right)^n}{n!} e^{-\int_0^T |g(t)| dt}$$

- Shifting: the parameter $c_Z$. 

How to obtain such a process $Z$? (Beskos and al. [1])

- Without loss of generality, suppose that $X$ is solution of

\[
\begin{align*}
\begin{cases}
  dX_t &= a(X_t)dt + dW_t \\
  X_0 &= x.
\end{cases}
\end{align*}
\]

(2)

(if $X$ has a non-constant diffusion coefficient $\sigma(.)$, make the change of
variables $Y_t = \eta(X_t)$ with $\eta(x) = \int_{x}^{x} \frac{1}{\sigma(u)} du$
)

- Denote by $(W_{t}^{x})_{t\in[0,T]}$ the process $(W_t + x)_{t\in[0,T]}$.

**Assumption 1 :**

\[
L_t = \exp \left[ \int_{0}^{t} a(W_{u}^{x})dW_{u}^{x} - \frac{1}{2} \int_{0}^{t} a^2(W_{u}^{x})du \right]
\]

is a martingale.
How to obtain such a process $Z$? (Beskos and al. [1])

- **Assumption 2**: $a$ is continuously differentiable.

Denote by $A$ the primitive of the drift $a$. By Itô’s lemma, one gets

$$L_T = \exp \left[ A(W_T^x) - A(x) - \frac{1}{2} \int_0^T a^2(W_t^x) + a'(W_t^x)dt \right].$$

- Consider the process $(Z_t)_{t \in [0,T]}$ distributed according to

$$Q_Z = \int_{\mathbb{R}} \mathcal{L} \left( (W_t^x)_{t \in [0,T]} \mid W_T^x = y \right) \rho(y) dy,$$

where $\rho$ is a positive density on the real line (another lever for variance reduction). By the Girsanov theorem,

$$C_0 = \mathbb{E} \left( f(Z_T) \psi(Z_T) \exp \left[ - \int_0^T \phi(Z_t) dt \right] \right) \quad (3)$$

where $\psi : z \mapsto \frac{e^{A(z)} - A(x) - (z-x)^2}{\sqrt{2\pi\rho(z)}}$ and $\phi : z \mapsto \frac{a^2(z) + a'(z)}{2}$. 
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In the Black & Scholes framework, under the risk-neutral measure

\[ \frac{dS_t}{S_t} = (r - \delta)dt + \sigma dW_t \]

so \( S_t = S_0e^{\sigma W_t + \gamma t} \) where \( \gamma = r - \delta - \frac{\sigma^2}{2} \).

Price of a continuous Asian option with pay-off \( f \):

\[ C_0 = \mathbb{E} \left( e^{-rT}f \left( \frac{1}{T} \int_0^T S_u \right) \right) \]

\( \Rightarrow \) No simple closed form solution

Numerical methods:

- Analytical approximations (Turnbull and Wakeman [15], Vorst [17], Levy [11] and more recently Lord [12]).
- PDE methods (see Vecer [16], Rogers and Shi [13], Ingersoll [8], Lelievre and Dubois [4]).
- Monte Carlo simulation methods (see Kemna and Vorst [9], Broadie and Glasserman [3], Fu and al. [6], Lapeyre and Temam [10]).
- Laplace transform inversion methods (see Geman and Yor [7]).
A priori, a two dimensional problem: let $\bar{S}_t = \frac{1}{t} \int_0^t S_u du$, then

$$
\begin{align*}
\frac{dS_t}{dt} &= S_t((r - \delta)dt + \sigma dW_t) \\
\frac{d\bar{S}_t}{dt} &= (-\frac{1}{t}\bar{S}_t + \frac{S_t}{t})dt
\end{align*}
$$

But, with a suitable change of variables (Rogers and Shi [13]), we can reduce the dimension to one:

$$
\xi_t = S_0 \int_0^t e^{\sigma(W_t-W_u)+\gamma(t-u)} du
$$

$$
\xi_T = \int_0^T S_0 e^{\sigma(W_T-W_{T-s})+\gamma s} ds
$$

and $\int_0^T S_u du$ have the same law so

$$
C_0 = \mathbb{E} \left( e^{-rT} f \left( \frac{1}{T} \xi_T \right) \right)
$$

where $(\xi_t)_{t \in [0,T]}$ is solution of

$$
\begin{align*}
\frac{d\xi_t}{dt} &= S_0 dt + \xi_t \left( \sigma dW_t + (\gamma + \frac{\sigma^2}{2}) dt \right) \\
\xi_0 &= 0
\end{align*}
$$
Need for a new change of variables:

\[
\xi_t = \frac{1}{t} S_0 \int_0^t e^{\sigma (W_t - W_u) + \gamma (t - u)} du
\]

\[
\xi_T = \frac{1}{T} \int_0^T S_0 e^{\sigma (W_T - W_T - s) + \gamma s} ds, \quad \text{and} \quad \frac{1}{T} \int_0^T S_u du \text{ have the same law so }
\]

\[
C_0 = \mathbb{E} \left( e^{-rT} f(\xi_T) \right)
\]

where \((\xi_t)_{t \in [0,T]}\) is solution of

\[
\begin{cases}
    d\xi_t &= \frac{\xi_0 - \xi_t}{t} dt + \xi_t \left( \sigma dW_t + (\gamma + \frac{\sigma^2}{2}) dt \right) \\
    \xi_0 &= S_0
\end{cases}
\]
\[ X_t = \log\left(\frac{\xi_t}{\xi_0}\right) \Rightarrow \begin{cases} 
  dX_t &= \sigma dW_t + \gamma dt + \frac{e^{-X_t}-1}{t}dt \\
  X_0 &= 0. 
\end{cases} \quad (4) \]

**Difficulty:** singularity of the drift term for \( t \to 0 \) prevents \( X \) from having an a.c law w.r.t the law of \( W \).
\[ X_t = \log\left(\frac{\xi_t}{\xi_0}\right) \Rightarrow \begin{cases} \quad dX_t = \sigma dW_t + \gamma dt + \frac{e^{-X_t}}{t} dt \\ \quad X_0 = 0. \end{cases} \] (4)

**Difficulty**: singularity of the drift term for \( t \to 0 \) prevents \( X \) from having an a.c law w.r.t the law of \( W \).

Consider instead

\[ dZ_t = \sigma dW_t + \gamma dt - \frac{Z_t}{t} dt; \quad Z_0 = X_0 = 0. \] (5)

**Lemma**

Existence and strong uniqueness hold for (4) and (5). Moreover,

\[ Z_t = \frac{\sigma}{t} \int_0^t s \, dW_s + \frac{\gamma}{2} t \] is a solution of (5).

\((Z_t)_{t \in [0,T]}\) is a Gaussian process and \( Z_T \sim \mathcal{N}\left(\frac{\gamma}{2} T, \frac{\sigma^2}{3} T\right)\).
Proposition

\[ L_t = \exp \left[ \int_0^t \frac{e^{-Z_s} - 1 + Z_s}{\sigma_s} dW_s - \frac{1}{2} \int_0^t \left( \frac{e^{-Z_s} - 1 + Z_s}{\sigma_s} \right)^2 ds \right] \]

is a martingale and hence \( C_0 = \mathbb{E} \left( e^{-rT} f(S_0e^{Z_T})L_T \right) \)

**Proof:** By the L.I.L of the Brownian motion, we show that \( \forall \epsilon > 0 \), there exists a (random) neighborhood of \( t = 0 \) for which

\[ |Z_t| \leq ct^{\frac{1}{2} - \epsilon} \text{ and } |X_t| \leq ct^{\frac{1}{2} - \epsilon} \]

hence, almost surely,

\[ \int_0^t \left( \frac{e^{-Z_s} - 1 + Z_s}{\sigma_s} \right)^2 ds < \infty \text{ and } \int_0^t \left( \frac{e^{-X_s} - 1 + X_s}{\sigma_s} \right)^2 ds < \infty. \]

Existence and strong uniqueness of the SDEs (4) and (5) permits to conclude (see Rydberg [14]).
Let $A(t, z) = \frac{1 - z + \frac{z^2}{2} - e^{-z}}{\sigma^2 t}$. By Itô’s lemma

$$A(T, Z_T) = \int_0^T e^{-Z_t} \frac{1 + Z_t}{\sigma^2 t} \, dZ_t - \int_0^T 1 - Z_t + \frac{Z_t^2}{2} - e^{-Z_t} \frac{dt}{\sigma^2 t^2} + \int_0^T \frac{1 - e^{-Z_t}}{2t} \, dt.$$

Finally, $C_0 = \mathbb{E} \left( e^{-rT} f(S_0 e^{Z_T}) e^{A(T, Z_T)} \exp \left[ \int_0^T \phi(t, Z_t) \, dt \right] \right)$ with

$$\phi(t, z) = \frac{1 - z + \frac{z^2}{2} - e^{-z}}{\sigma^2 t^2} - \frac{1 - e^{-z}}{2t} - \frac{e^{-z} - 1 + z}{\sigma^2 t} \left( \frac{e^{-z} - 1 - z}{2t} + \gamma \right).$$
A first conjecture

In order to be able to deal with both calls and puts, we need the following integrability condition (I.C)

**Conjecture**

\[
\mathbb{E} \left( e^{A(T,Z_T)-rT}(e^{Z_T} + 1)e^{\int_0^T |\phi(t,Z_t)|dt} \right) < \infty.
\]

For a call, this implies that

\[
C_0 = \mathbb{E} \left( e^{A(T,Z_T)-rT}(S_0e^{Z_T} - K) + \frac{1}{p(N)N!} \prod_{i=1}^N \frac{\phi(U_i, Z_{U_i})}{q(U_i)} \right)
\]

with well chosen probability distributions \(p\) and \(q\).
Choice of the distributions $p$ and $q$

We need square integrability in order to construct confidence intervals.

$$
E\left( e^{2A(T,Z_T)−2rT} f^2(S_0 e^{Z_T}) \frac{\left( \int_0^T \frac{\phi^2(t,Z_t)}{q(t)} dt \right)^N}{p(N)^2 (N!)^2} \right) < \infty ?
$$

False for the naive choice of a uniform distribution for $q$:

**Lemma**

\[ \forall \epsilon > 0, \text{ we have a.s. } \phi(t, Z_t) - \frac{2Z_t^3}{3\sigma^2 t^2} + \frac{Z_t}{2t} = O(t^{-\epsilon}). \]

Therefore, we have

$$
\int_0^T \frac{\phi^2(t,Z_t)}{t^a} dt < \infty \text{ a.s. if and only if } a < 0.
$$
Variance reduction

With \( p = \mathcal{P}(c PT) \) and \( q(t) = \frac{1}{2\sqrt{t} \sqrt{T}} \mathbb{1}_{[0, T]}(t) \) (since \( \phi \sim \frac{1}{\sqrt{t}} \) near 0), our estimator writes

\[
\delta = \frac{1}{m} \sum_{j=1}^{m} e^{A(T,Z_T^j) - rT} (S_0 e^{Z_T^j} - K) + e^{c PT - c Z_T^j T} \prod_{i=1}^{N^j} \frac{2 \sqrt{U_{i,k}} \left( c_{Z_i} - \phi(U_{i,k}, Z_{U_{i,k}}) \right)}{c_p \sqrt{T}}
\]

- **Conditioning**: for every simulated trajectory \( Z^j \), we compute

\[
\frac{1}{n} \sum_{k=1}^{n} \prod_{i=1}^{N^j} \frac{2 \sqrt{U_{i,k}} \left( c_{Z_i} - \phi(U_{i,k}, Z_{U_{i,k}}) \right)}{c_p \sqrt{T}}
\]

instead of

\[
\prod_{i=1}^{N^j} \frac{2 \sqrt{U_{i,k}} \left( c_{Z_i} - \phi(U_{i,k}, Z_{U_{i,k}}) \right)}{c_p \sqrt{T}}
\]

- **Control variate**: we can use \( e^{-rT} (S_0 e^{Z_T} - K) \) as a control variate since \( Z_T \sim \mathcal{N}\left( \frac{\gamma}{2} T, \frac{\sigma^2 T}{3} \right) \).
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Numerical test of the conjectures

\[
\frac{1}{m} \sum_{j=1}^{m} e^{A(T,Z_T^j) - rT} (S_0 e^{Z_T^j} + 1) e^{c_p T} \prod_{i=1}^{N_j} 2 \sqrt{U_i^j} \frac{\phi(U_i^j, Z_{U_i^j}^j)}{c_p \sqrt{T}}
\]  

(6)

**Fig.**: Evaluation of (6) with respect to the number \( m \) of simulations
Numerical test of the conjectures

\[
\frac{1}{m} \sum_{j=1}^{m} e^{2A(T, Z^j_T)} - 2rT (S_0 e^{Z^j_T} + 1)^2 e^{2c_pT} \prod_{i=1}^{N^j} \frac{4U_i^j \phi^2(U_i^j, Z_i^j U_i^j)}{c_p^2 T}
\]  

(7)

**Fig.**: Evaluation of (7) with respect to the number \( m \) of simulations
Comparison with a standard Monte Carlo method

Set of parameters:
- \( S_0 = K = 100 \)
- \( r = 0.1 \)
- \( \sigma = 0.2 \)
- \( T = 1 \)

<table>
<thead>
<tr>
<th>Method</th>
<th>Price</th>
<th>L.C.I at 95%</th>
<th>N</th>
<th>CPU</th>
</tr>
</thead>
<tbody>
<tr>
<td>E.C.E std</td>
<td>7.035</td>
<td>0.035</td>
<td>(2 \times 10^5)</td>
<td>(\sim 1)s</td>
</tr>
<tr>
<td>E.C.E opt</td>
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<td>0.005</td>
<td>(2 \times 10^5)</td>
<td>(\sim 1)s</td>
</tr>
<tr>
<td>MC std (Trap)</td>
<td>7.051</td>
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<td>(\sim 1)s</td>
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<tr>
<td>MC opt (Trap+KV)</td>
<td>7.041</td>
<td>0.002</td>
<td>(10^5)</td>
<td>(\sim 1)s</td>
</tr>
</tbody>
</table>

Tab.: Asian call price with different MC methods. For the standard E.C.E (without variance reduction), we took \( c_p = 1 \). For the optimized E.C.E, we took \( c_p = c_T = \frac{1}{2T} \) and \( n = 5 \). For MC, the number of time steps is 20.
Comparison with a standard Monte Carlo method

Set of parameters:
- \( S_0 = K = 100 \)
- \( r = 0.1 \)
- \( \sigma = 0.2 \)
- \( T = 1 \)

<table>
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</tr>
</thead>
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<tr>
<td>E.C.E opt</td>
<td>7.0404</td>
<td>0.0001</td>
<td>( 7 \times 10^6 )</td>
<td>( \sim 15s )</td>
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<tr>
<td>MC opt (Trap+KV)</td>
<td>7.0401</td>
<td>0.0008</td>
<td>( 10^6 )</td>
<td>( \sim 15s )</td>
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**Tab.:** Asian call price with different MC methods. For the optimized E.C.E, we took \( c_p = c_T = \frac{1}{2T} \) and \( n = 5 \). For Trap+KV, the number of time steps is 100.
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Conclusion

Pros

- A MC price of an Asian option that is not prone to discretization bias.
  \[ \rightarrow \text{A reliable benchmark} \]
- A competitive method if high precision is required.
- A competitive MC method for pricing Asian like options with pay-off
  \[ \alpha S_T + \beta \int_0^T S_u du, \alpha \neq 0. \]

Cons

- Less competitive than an optimized Monte Carlo (Lapeyre and Temam [10]) for usual precision levels (any other possible variance reduction method?).
- No theoretical justification of the integrability conjectures.
Merci !


References II

Paul Fearnhead, O. Papaspiliopoulos, and Gareth O. Roberts. 
Particle filters for partially observed diffusions. 

Pricing continuous asian options : a comparison of monte carlo and laplace transform inversion methods. 
*Journal of Computational Finance, 2(2), 1999.*

H. Geman and M. Yor. 
Bessel processes, asian option and perpetuities. 
*Mathematical Finance, 3(4), 1993.*

J.E. Ingersoll. 
*Theory of Financial Decision Making.* 
References III

A. Kemna and A. Vorst.
A pricing method for options based on average asset values.

B. Lapeyre and E. Temam.
Competitive Monte Carlo methods for pricing asian options.

E. Levy.
Pricing european average rate currency options.

R. Lord.
Partially exact and bounded approximations for arithmetic Asian options.
L. C. G. Rogers and Z. Shi.
The value of an Asian option. 

T. H. Rydberg.
A note on the existence of unique equivalent martingale measures in a markovian setting. 

S. Turnbull and L. Wakeman.
A quick algorithm for pricing european average options. 

J. Vecer.
A new pde approach for pricing arithmetic asian options. 
References V

T. Vorst.
Prices and hedge ratios of average exchange rate options.

W. Wagner.
Unbiased Monte Carlo evaluation of certain functional integrals.