Asymptotic analysis of hedging errors in models with jumps

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Hedging problem: introduction

We have a financial asset with price S_t modeled by a stochastic process. For example,

$$dS_t = rS_t dt + \sigma S_t dW_t.$$

An option on this asset is a *derivative* product depending on S_t. Its payoff at a future date T is modeled by a random variable H_T. For example,

$$H_T = max(S_T - K, 0).$$

Hedging problem: approach H_T by a dynamic portfolio containing bonds and stock S_t

$$H_T \approx V_T = \int_0^T \phi_t^0 dB_t + \int_0^T \phi_t dS_t.$$

Example: delta-hedging in diffusion models

Suppose that stock price S_t is a diffusion process:

$$\frac{dS_t}{S_t} = \mu(t, S_t)dt + \sigma(t, S_t)dW_t$$

Consider a European option with terminal payoff $H_T = h(S_T)$. Its price has the form $C_t = C(t, S_t)$.

Then there exists a hedging portfolio which replicates H_T exactly:

$$H_T = \int_0^T \phi_t dS_t$$

(we assume for simplicity $r = 0$)

with $\phi_t = \frac{\partial C}{\partial S}(t, S_t)$.

Hedging in incomplete markets

- Incomplete market: exact replication impossible.
- Hedging is now an approximation problem.
- Industry practice: sensitivities to risk factors

$$\text{Delta} = \frac{\partial C(t, S_t)}{\partial S} : \quad \text{infinitesimal moves, hedge with stock}$$

$$\text{Gamma} = \frac{\partial^2 C(t, S_t)}{\partial S^2} : \quad \text{bigger moves; hedge with liquid options}$$

- Quadratic hedging: control the residual error

$$\min_{\phi} E\left(c + \int_0^T \phi_t dS_t - H_T\right)^2$$

All these strategies require a continuously rebalanced portfolio.

Discrete hedging

- Continuous rebalancing is unfeasible: in practice, the strategy ϕ_t is replaced with a discrete strategy, leading to the hedging error of the "second type": error of approximating the continuous portfolio with a discrete one.
- The simplest choice is $\phi_t^n := \phi_{h[t/h]}$, $h = T/n$.
- This discretization error has only been studied in the case of continuous processes.

Discrete hedging: the complete market case

- Bertsimas, Kogan and Lo '98 introduced an *asymptotic approach* allowing to study discrete hedging in continuous time.

Suppose

$$\frac{dS_t}{S_t} = \mu(t, S_t)dt + \sigma(t, S_t)dW_t$$

and we want to hedge a European option with payoff $h(S_T)$ using delta-hedging $\phi_t = \frac{\partial C}{\partial S}$.

CLT for hedging error

The discrete hedging error is defined by

$$\varepsilon_T^n = h(S_T) - \int_0^T \phi_t^n dS_t$$

Then $\varepsilon_T^n \to 0$ but the renormalized error $\sqrt{n}\varepsilon_T^n$ converges to

$$\sqrt{\frac{T}{2}} \int_0^T \frac{\partial^2 C}{\partial S^2} S_t^2 \sigma_t^2 dW_t^*,$$

where W^* is a Brownian motion independent of W.

- Hedging error decays as $\sqrt{h}$.
- It is orthogonal to the stock price.
- The amplitude is determined by the gamma $\frac{\partial^2 C}{\partial S^2}$

Intuition

In complete market,

$$\varepsilon_T^n = \int_0^T (\phi_t - \phi_t^n) dS_t$$

Let $S_t = W_t$ and consider the renormalized error over one hedging interval:

$$\frac{1}{\sqrt{h}} \int_0^h \left(\frac{\partial C}{\partial S}(W_t) - \frac{\partial C}{\partial S}(0)\right) dW_t \approx \frac{1}{\sqrt{2}} \frac{\partial^2 C}{\partial S^2} \frac{1}{\sqrt{2h}} (W_h^2 - h)$$

The random variable $\frac{1}{\sqrt{2h}}(W_h^2 - h)$ has mean zero, variance h and is uncorrelated with W_h.

Approximating hedging portfolios

Hayashi and Mykland '05 interpreted the discrete hedging error as the error of approximating the "ideal" hedging portfolio $\int_0^T \phi_t dS_t$ with a feasible hedging portfolio $\int_0^T \phi_t^n dS_t$

- This makes sense in incomplete markets

Suppose ϕ and S are Itô processes:
$d\phi_t = \tilde{\mu}_t dt + \tilde{\sigma}_t dW_t$ and $dS_t = \mu_t dt + \sigma_t dW_t$. Then

$$\sqrt{n}\varepsilon_t^n \Rightarrow \sqrt{\frac{T}{2}} \int_0^t \tilde{\sigma}_s \sigma_s dW_s^*,$$

$$\text{where} \quad \varepsilon_t^n := \int_0^t (\phi_s - \phi_s^n) dS_s.$$

- Weak convergence of processes in the Skorokhod topology on the space $\mathbb{D}$ of càdlàg functions

Discrete hedging in presence of jumps

The idea of approximating stochastic integrals goes back to Rootzen (80)
More recently, results by Geiss (02), (06), (07) but all authors work with continuous processes

Our idea: study the discretization error

$$\varepsilon_t^n := \int_0^t (\phi_{s-} - \phi_{s-}^n) dS_s$$

in presence of jumps in the underlying and the hedging strategy.

- Some tools are available in the study of the approximation error of the Lévy-driven Euler scheme by Jacod and Protter (98)

Model setup: Lévy-Itô processes

$$X_t = X_0 + \int_0^t \mu_s ds + \int_0^t \sigma_s dW_s + \int_0^t \int_{|z|\leq 1} \gamma_s(z)\tilde{J}(ds \times dz) + \int_0^t \int_{|z|>1} \gamma_s(z)J(ds \times dz).$$

- J: Poisson random measure with intensity $dt \times \nu$
- μ and σ are càdlàg $(\mathcal{F}_t)$-adapted
- γ: $\Omega \times [0, T] \times \mathbb{R} \to \mathbb{R}$ is such that $(\omega, z) \mapsto \gamma_t(z)$ is $\mathcal{F}_t \times \mathcal{B}(\mathbb{R})$-measurable $\forall t$ and $t \to \gamma_t(z)$ is càglàd $\forall \omega, z$;

$$\gamma_t(z)^2 \leq A_t \rho(z), \qquad \int_{|z|\leq 1} \rho(z)\nu(dz) < \infty$$

with ρ positive deterministic and A càglàd $(\mathcal{F}_t)$-adapted.

Model setup

- The stock price S is a Lévy-Itô process with coefficients μ, σ, γ;
- The continuous-time strategy ϕ is a Lévy-Itô process (driven by the same W and J) with coefficients $\tilde{\mu}, \tilde{\sigma}, \tilde{\gamma}$.
- The agent uses the discrete-time strategy $\phi_t^n := \phi_{h[t/h]}$ instead of the continuous-time strategy ϕ_t.

Main result

The discretization error satisfies

$$\sqrt{n}\varepsilon_t^n \to \sqrt{\frac{T}{2}}\int_0^t \sigma_s\tilde{\sigma}_s dW_s^* + \sqrt{T}\sum_{i:T_i\leq t}\Delta\phi_{T_i}\sqrt{\zeta_i}\xi_i\sigma_{T_i}$$
$$+\sqrt{T}\sum_{i:T_i\leq t}\Delta S_{T_i}\sqrt{1-\zeta_i}\xi_i'\tilde{\sigma}_{T_i-}.$$

W^* is a standard BM independent from W and J,
$(\xi_k)_{k\geq 1}$ and $(\xi_k')_{k\geq 1}$ are two sequences of independent $N(0,1)$,
$(\zeta_k)_{k\geq 1}$ is sequence of independent $U([0,1])$
$(T_i)_{i\geq 1}$ are the jump times of J enumerated in any order.

The normalizing sequence

The normalizing factor need not be equal to $\sqrt{n}$.
Suppose ϕ and S move only by finite-intensity jumps. If there is only one jump between t_k and t_{k+1},

$$\int_{t_k}^{t_{k+1}} \phi_{t-} dS_t = \phi_{T_i -} \Delta S_{T_i} = \phi_{t_k} \Delta S_{T_i} = \int_{t_k}^{t_{k+1}} \phi^n_{t-} dS_t$$

Therefore $P[\varepsilon^n_t \neq 0] = O(1/n)$ and

$$n^\alpha \varepsilon^n_t \longrightarrow 0$$

in probability $\forall \alpha$.
More generally, if S and ϕ are Lévy-Itô processes without diffusion parts,

$$\sqrt{n} \varepsilon^n_t \longrightarrow 0$$

in probability uniformly on t.

Application: delta-hedging in a Lévy market

$$S_t = S_0 e^{X_t}, \quad X_t = bt + \sigma W_t + \int_0^t \int zJ(ds \times dz)$$

$$C(t,S) = E^Q[H(Se^{X_{T-t}})], \quad \phi_t = \frac{\partial C}{\partial S}(t, S_t)$$

Suppose

- The Lévy measure is finite and has a regular density (e.g. Merton model).
- The payoff function H is piecewise smooth with a finite number of discontinuities.

Application: delta-hedging in a Lévy market

Apply the Itô formula to get the decomposition for ϕ:

$$d\phi_t = d\frac{\partial C(t,S_t)}{\partial S} = \left\{ \frac{\partial^2 C}{\partial t \partial S} + (b+\sigma^2/2)\frac{\partial^2 C}{\partial S^2} S_t + \frac{\sigma^2}{2}\frac{\partial^3 C}{\partial S^3} S_t^2 \right\} dt$$

$$+\sigma \frac{\partial^2 C}{\partial S^2} S_t dW_t + \int_{\mathbb{R}} \left(\frac{\partial C}{\partial S}(t, S_{t-}e^z) - \frac{\partial C}{\partial S} C(t, S_{t-}) \right) J(dt \times dz)$$

Under the hypotheses on H and ν it can be shown that the coefficients do not explose in T: almost all trajectories end in a point where H is smooth.

Application: delta-hedging in a Lévy market

The main result then implies $\sqrt{n}\varepsilon_t^n \to Z_t$ with

$$Z_t = \sqrt{\frac{T}{2}} \int_0^t \sigma^2 S_s^2 \frac{\partial^2 C}{\partial S^2} dW_s^* + \sqrt{T} \sum \Delta \frac{\partial C}{\partial S} \sqrt{\zeta_i} \xi_i \sigma S_s$$
$$+ \sqrt{T} \sum \Delta S_s \sqrt{1 - \zeta_i} \xi_i' \sigma S_{s-} \frac{\partial^2 C}{\partial S^2}(s, S_{s-})$$

Application: risk of a hedged option position

If $E[Z_t^2] < \infty$, we can estimate the risk of a hedged option position using

$$P[|\varepsilon_t^n| \geq \delta] \leq \frac{1}{\delta\sqrt{n}} E[Z_t^2]^{1/2}$$

with (small jump size approximation)

$$E[Z_t^2] \approx \frac{T}{2} \int_0^t E\left[S_s^4 \left(\frac{\partial^2 C}{\partial S^2}\right)^2\right] (\sigma^4 + \sigma^2 \int (e^z - 1)^2 (e^{2z} + 1) \nu(dx)).$$

This should be compared to the MSE from market incompleteness:

$$E[\epsilon_t^2] \approx \frac{1}{4} \int_0^t E\left[S_s^4 \left(\frac{\partial^2 C}{\partial S^2}\right)^2\right] \int (e^z - 1)^4 \nu(dx).$$

Merci!

Idea of the proof

Main tools:

- If (X^n) and (Y^n) are two sequences of processes such that

$$\sup_t |X^n_t - Y^n_t| \to 0 \quad \text{in probability}$$

and $X^n \to X$ weakly then $Y^n \to X$ weakly.

- Let $(\Omega_m)_{m\geq 1}$ be a sequence of subsets of Ω with

$$\lim_m P(\Omega_m) = 1$$

If, for every m, $X_n 1_{\Omega_m} \to X 1_{\Omega_m}$ weakly, then $X_n \to X$ weakly.

Allows to reduce to bounded coefficients and bounded jumps.

Idea of the proof

Step 1 Remove the big jumps

Step 2 Remove the small jumps

Step 3 Now we can write

$$S_t = S_0 + S_t^d + S_t^c + S_t^j$$

$$S_t^d = \int_0^t \left(\mu_s + \int \gamma_s(z)\nu(dz) \right) ds$$

$$S_t^c = \int_0^t \sigma_s dW_s$$

$$S_t^j = \int_0^t \int \gamma_s(z) J(ds \times dz)$$

and $\phi_t = \phi_0 + \phi_t^d + \phi_t^c + \phi_t^j$.

Idea of the proof

The leading terms in the hedging error are

1. $$\sqrt{n}\int_0^t (\phi_s^c - \phi_s^{c,n})dS_s^c \quad \to \quad \sqrt{\frac{T}{2}}\int_0^t \sigma_s\tilde{\sigma}_s dW_s^*$$
   as in Bertsimas et al. '98

2. $$\sqrt{n}\int_0^t (\phi_s^j - \phi_s^{j,n})dS_s^c$$ (see next slides)

3. $$\sqrt{n}\int_0^t (\phi_s^c - \phi_s^{c,n})dS_s^j$$ (see next slides)

Idea of the proof

Notation: if $T_i \in (t_k, t_{k+1}]$ then $t_k = \theta(T_i)$ and $t_{k+1} = \psi(T_i)$.

$$2. \quad \sqrt{n} \int_0^t (\phi_s^j - \phi_s^{j,n}) dS_s^c \quad \approx \quad \sqrt{n} \sum_{i: T_i \le t} \Delta\phi_{T_i} \int_{T_i}^{\psi(T_i)} \sigma_s dW_s$$

(the two processes differ if there exist $(t_k, t_{k+1}]$ with ≥ 2 jumps: an event with probability $O(1/n)$)

$$\approx \sum_{i: T_i \le t} \Delta\phi_{T_i} \sigma_{T_i} \sqrt{n} \int_{T_i}^{\psi(T_i)} dW_s \quad \to \quad \sqrt{T} \sum_{i: T_i \le t} \Delta\phi_{T_i} \sigma_{T_i} \sqrt{\zeta_i} \xi_i$$

since

$$\sqrt{n}(W_{\psi(T_i)} - W_{T_i}) \overset{d}{=} \sqrt{n(\psi(T_i) - T_i)} W_1 \overset{d}{\approx} \sqrt{nU([0, T/n])}\, N(0,1)$$

Idea of the proof

Similarly,

$$
\begin{aligned}
3. \quad \sqrt{n}\int_0^t (\phi_s^c - \phi_s^{c,n})dS_s^j &= \sqrt{n}\sum_{i:T_i\le t} \Delta S_{T_i}\int_{\theta(T_i)}^{T_i} \tilde{\sigma}_s dW_s \\
&\approx \sum_{i:T_i\le t} \Delta S_{T_i}\tilde{\sigma}_{\theta(T_i)}\sqrt{n}\int_{\theta(T_i)}^{T_i} dW_s \\
&\to \sqrt{T}\sum_{i:T_i\le t} \Delta S_{T_i}\tilde{\sigma}_{T_i-}\sqrt{1-\zeta_i}\xi_i'
\end{aligned}
$$

since

$$
\sqrt{n}(W_{T_i} - W_{\theta(T_i)}) \overset{d}{\approx} \sqrt{n(T_i - \theta(T_i))}W_1
$$

and

$$
T_i - \theta(T_i) = T/n - (\psi(T_i) - T_i))
$$